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On the Indefinite $\backslash(A \backslash)$ -Integral

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Abstract

Full Text

MATHEMATICS

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On the Indefinite A -Integral

(Presented by Academician A. N. Kolmogorov on 28 VI 1960)

It is known ⁽¹⁾ that if $f(x) \in L(-\pi, \pi)$, then for almost all $x \in [-\pi, \pi]$ the function

$$\bar{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \operatorname{ctg} \frac{1}{2}(t-x) dt,$$

is defined, where the integral is taken in the sense of the Cauchy principal value. This fact was established by N. N. Luzin for functions of class L^2 and by I. I. Privalov for functions of class L . The function $\bar{f}(x)$, generally speaking, is not summable on $[-\pi, \pi]$, so that in order to integrate it one must introduce a new concept of integral.

A function $f(x)$ is called A -integrable on $[a, b]$ if

$$mE\{x, x \in [a, b], |f(x)| > n\} = o\left(\frac{1}{n}\right) \quad (1)$$

and there exists

$$\lim_{n \rightarrow \infty} \int_a^b [f(x)]^n dx, \quad \text{where} \quad [f(x)]^n = \begin{cases} f(x), & \text{if } |f(x)| \leq n, \\ 0, & \text{if } |f(x)| > n. \end{cases} \quad (2)$$

The value of this limit is called the definite A -integral of $f(x)$ on $[a, b]$.

As Titchmarsh showed ⁽⁵⁾, every function conjugate to a summable one is A -integrable on $[-\pi, \pi]$ and, moreover, on the interval $[a, b]$ for almost all $a, b \in [-\pi, \pi]$. It is also shown there that, without condition (1), an integral defined only by the limit (2) may be nonadditive. Titchmarsh did not introduce this condition specially, but all his investigations concern functions conjugate to summable ones, for which this condition is satisfied, as was earlier shown by A. N. Kolmogorov ⁽²⁾. A complete definition of the A -integral as the limit (2) under condition (1) was given by A. N. Kolmogorov in the probabilistic form of generalized mathematical expectation ⁽³⁾, Ch. 6). Later the A -integral was considered by J. S. Ochan, in whose work ⁽⁴⁾, part 4) A. N. Kolmogorov's definition is presented from the point of view of function theory. It is also

shown there that the sets of A -integrable functions and of functions integrable in the Denjoy sense, both narrow and broad, partially intersect. A number of works by P. L. Ulyanov ⁽⁶⁻⁹⁾ are devoted to questions of A -integration and the application of the A -integral in the theory of trigonometric series. In paper ⁽⁸⁾ it is shown, in particular, that the existence of the A -integral of $f(x)$ on an interval $[a, b]$ does not guarantee the existence of this integral on an interval $[a', b'] \in [a, b]$, where a' and b' range over a set of positive measure. Thus, having the possibility of determining the value of the definite A -integral of $f(x)$ on $[a, b]$, one cannot, generally speaking, define the indefinite A -integral of $f(x)$ on $[a, b]$ as the function

$$A(x) = (A) \int_a^x f(t) dt.$$

It is known that if $f(x) \in L(a, b)$, then $f(x)$ is A -integrable on $[a, b]$ and the two integrals coincide; consequently, for any summable function on $[a, b]$ the indefinite A -integral exists and, for all $x \in [a, b]$,

$$(L) \int_a^x f(t) dt = (A) \int_a^x f(t) dt.$$

If in the right-hand side of this equality the Lebesgue integral is replaced by the Denjoy integral, then the equality loses its meaning for some functions, since a D -integrable function, as was indicated above, is not always A -integrable. Below we give theorems showing that this equality may fail even under the condition of the simultaneous existence of the A -integral and the Denjoy integral, even in the narrow sense.

We shall say that the A -integral contradicts the Denjoy integral at a point $x \in [a, b]$ if $f(x)$ is A - and D -integrable on $[a, x]$ and

$$(D) \int_a^x f(t) dt \neq (A) \int_a^x f(t) dt.$$

As follows from the results of Titchmarsh and A. G. Dzvaršeišvili ⁽¹⁰⁾, for functions conjugate to summable and Denjoy-integrable functions, a contradiction can occur only on a set of measure zero.

We shall say that a function $f(x)$ has an indefinite A -integral on $[a, b]$ if $f(x)$ is A -integrable on $[a, x]$ for all $x \in [a, b]$.

From the existence of the indefinite A -integral its continuity does not follow, as the following theorem shows.

Theorem 1. There exists a function $f(x)$, $x \in [0, 1]$, integrable on $[0, 1]$ in the sense of an improper Lebesgue integral, for which the indefinite A -integral exists on $[0, 1]$ and is discontinuous at the point $x = 1$.

It is obvious that for functions integrable in the sense of an improper Lebesgue integral on some interval $[a, b]$, the discontinuity of the indefinite A -integral on this interval is a necessary and sufficient condition for a contradiction of these two integrals at some point $x \in [a, b]$. However, in general, for the Denjoy integral the discontinuity of the indefinite A -integral is not a necessary condition for contradiction, as the following theorem shows.

Theorem 2. There exists a function $f(x)$, $x \in [0, 1]$, possessing the following properties:

- a) $f(x)$ is Denjoy integrable on $[0, 1]$ and is the exact derivative of its indefinite D -integral;
- b) $f(x)$ has a continuous indefinite A -integral on $[0, 1]$

$$A(x) = (A) \int_0^x f(t) dt;$$

c)

$$A(x) \neq (D) \int_0^x f(t) dt \quad (x \in P, mP > 0),$$

where $A(x)$ either does not have the N -property, or does not have an asymptotic derivative on a set of positive measure, or has an almost everywhere derivative, but not equal to $f(x)$ on a set of positive measure.

The question arises: how broad is the set of functions $A(x)$ that are a continuous indefinite integral of some function? The answer to this question is given by the following theorem.

Theorem 3. Let $F(x)$, $x \in [0, 1]$, be an arbitrary continuous function such that $F(0) = 0$. Then there exists a function $f(x)$ for which there exists an indefinite A -integral on $[0, 1]$,

$$A(x) = (A) \int_0^x f(t) dt,$$

and

$$A(x) = F(x) \quad (x \in [0, 1]),$$

where the sequence of functions

$$A_n(x) = (A) \int_0^x |f(t)|^n dt$$

(see (2)) converges uniformly on $[0, 1]$ to $A(x)$ as $n \rightarrow \infty$.

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