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Abstract

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PS-ISOMORPHISM OF AN ORDERED GROUP

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In this paper it is proved that an ordered group is determined by the structure of its subsemigroups.

Let a one-to-one correspondence φ be established between the elements of two groups G and G^{φ}. To an element $g \in G$ there will correspond an element $\varphi(g) \in G^{\varphi}$. Elements a, b of the group G will be called directly (respectively, inversely) φ-parallel if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

(respectively, if

$$\varphi(ab) = \varphi(b)\varphi(a)).$$

If a, b are either directly or inversely φ-parallel, then they are called φ-parallel. Finally, if the elements a, b are (directly, inversely) φ-parallel and the elements b, a are (directly, inversely) φ-parallel, then they are called mutually (directly, inversely) φ-parallel.

Elements a and b of the group G will be called separable by a subsemigroup H if the element b belongs to H, the element a belongs to the normalizer $N(H)$ of the subsemigroup H, and the intersection of the subsemigroup H with the cyclic subgroup $\{a\}$ is equal to the identity.

Elements a and b of the group G will be called separable if there exists a subsemigroup H such that a and b are separable by the subsemigroup H.

The product ab of elements a and b of the group G without torsion, having no other expression as a product of a finite number of positive powers of the elements a and b, will be called unambiguous. In the contrary case the product ab is called ambiguous.

Let φ be an isomorphism of the structures of subsemigroups (a PS-isomorphism (1)) of the groups G and G^{φ}. If A is a subsemigroup in G, then A^{φ} is its image in G^{φ} under the PS-isomorphism φ. $\{A, B\}$ denotes the subsemigroup generated by the subsemigroups A and B. In what follows we consider a group G without torsion. It is known that in this case the group G^{φ} is also without torsion and that between the elements of the groups G and G^{φ} one can establish a one-to-one correspondence φ. To an element $g \in G$ there will correspond an element $\varphi(g) \in G^{\varphi}$, and

$$\{g\}^{\varphi} = \{\varphi(g)\}.$$

For permutable elements a and b from G it is known:

$$\varphi(ab) = \varphi(ba) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a),$$

i.e. they are mutually φ-parallel (^{2}).

A PS-isomorphism φ of a torsion-free group G onto a group G^φ, under which any two separable elements a, b are mutually φ-parallel, will be called proper.

Lemma 1. Let the elements a and b of the torsion-free group G be nonpermutable and mutually φ-parallel. Then, if a, b are directly φ-parallel, then b, a are also directly φ-parallel.

Lemma 2. Let G be a torsion-free group. Let the elements a, b of the group G be separable by a commutative subsemigroup H. Then the elements $\varphi(a), \varphi(b)$ of the group G^φ are separable by the subsemigroup H^φ, and the correspondence φ between the elements of the groups G and G^φ is a group isomorphism or anti-isomorphism of the subgroups H and H^φ.

Lemma 3. Let a and b be noncommuting isolated elements of an R-group G, and let at least one of the products ab or ba be unambiguous. Then a, b are isolated by a locally cyclic subgroup H.

On the basis of Lemmas 2 and 3 the following theorem is proved.

Theorem 1. Let G be an R-group and let H be an isolated normal divisor in G. Then H^φ is an isolated normal divisor in G^φ.

Lemma 4. Let φ be a proper PS-isomorphism of the groups G and G^φ; let H and F be isolated normal divisors in G and $H \subset F$; let x and y be noncommuting elements, with $x \in H$, $y \in F \setminus H$. Then $\varphi(xy) = \varphi(x)\varphi(y)$ implies $\varphi(xz) = \varphi(x)\varphi(z)$ for any $z \in G \setminus F$.

Proof. It is enough to consider noncommuting elements x and z. Let $\varphi(xz) = \varphi(z)\varphi(x)$. 1) First we prove the lemma for any element $z \in G \setminus F$ noncommuting with y. Using the fact that the PS-isomorphism φ is proper, and Lemma 1, in exactly the same way as in (^{3}), item (3), item (5) case 1 and item (6), we obtain a contradiction. 2) Now suppose $zy = yz$. The equality $\varphi(xy) = \varphi(x)\varphi(y)$, by item (3) of (^{3}) and Lemma 1, implies

$$\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x) = \varphi(x)\varphi(yx).$$

The element yx does not commute with z, since z does not commute with x. The element $yx \in F \setminus H$. Therefore the proof is the same as in item 1), where in place of the element y one must take the element yx.

Lemma 5. Let φ be a proper PS-isomorphism of the groups G and G^φ; let H and F be isolated normal divisors in G and $H \subset F$; let f and g be noncommuting elements, with $g \in F$, $f \in F \setminus H$. Then $\varphi(fg) = \varphi(f)\varphi(g)$ implies $\varphi(fh) = \varphi(f)\varphi(h)$ for any element $h \in H$ noncommuting with g.

Proof. It is enough to consider the case when f and h do not commute. The proof is literally the same as in item 1) of Lemma 4. In this proof one must put $x = f,\ y = g,\ z = h$.

Theorem 2. Let φ be a proper PS-isomorphism of the groups G and G^{φ}; let H and F be proper isolated normal divisors in G and $H \subset F$. Then either any two elements $h_1, h_2 \in H$ are directly φ-parallel, or oppositely φ-parallel.

Proof. 1) Suppose that for any two elements x and y such that $x \in H,\ y \in F \setminus H$, the equality $\varphi(yx) = \varphi(x)\varphi(y)$ is valid; then we get:

$$\varphi(h_1h_2)\varphi(y) = \varphi(yh_1h_2) = \varphi(h_2)\varphi(yh_1) = \varphi(h_2)\varphi(h_1)\varphi(y),$$

and the theorem is valid. 2) Suppose now that there is at least one pair of noncommuting elements x, y such that $x \in H,\ y \in F \setminus H$, for which $\varphi(yx) = \varphi(y)\varphi(x)$. Let z be any element such that $z \in F$ and $zy \neq yz$; then $\varphi(zy) = \varphi(z)\varphi(y)$. Indeed, if $\varphi(yzx) = \varphi(y)\varphi(zx)$, then, by Lemmas 1 and 4,

$$\varphi(yz)\varphi(x) = \varphi(yzx) = \varphi(y)\varphi(zx) = \varphi(y)\varphi(z)\varphi(x),$$

and after cancellation, by Lemma 1, we obtain $\varphi(zy) = \varphi(z)\varphi(y)$. If $\varphi(yzx) = \varphi(zx)\varphi(y)$, then, by Lemmas 1 and 4,

$$\varphi(z)\varphi(x)\varphi(y) = \varphi(zx)\varphi(y) = \varphi(yzx)\varphi(yz)\varphi(x) = \varphi(y)\varphi(z)\varphi(x).$$

(The last equality, in view of the fact that $\varphi(yz)\varphi(x) = \varphi(z)\varphi(y)\varphi(x)$, would imply $\varphi(z)\varphi(x)\varphi(y) = \varphi(z)\varphi(y)\varphi(x)$ and commutativity of x and y, which is false.) Again after cancellation, by Lemma 1, we obtain $\varphi(zy) = \varphi(z)\varphi(y)$. Put in Lemma 5 $g = z,\ f = y$ and $h = x$, if $zy \neq yz$, and put $g = z,\ f = yx$ and $h = x$, if $zy = yz$. We obtain, for any element x from H noncommuting with z, $\varphi(yx) = \varphi(y)\varphi(x)$ (or $\varphi(yxx) = \varphi(yx)\varphi(x)$). Now, by Lemmas 1 and 4, for any two noncommuting z and x such that $x \in H,\ z \in F$, we obtain $\varphi(xz) = \varphi(x)\varphi(z)$. Therefore for any elements x and z, if $x \in H,\ z \in F$, then $\varphi(xz) = \varphi(x)\varphi(z)$. Further, in the same way as in item 1), we prove that for any h_1, h_2 from H one has

$$\varphi(h_1h_2) = \varphi(h_1)\varphi(h_2).$$

Let A be a semigroup from the group G; let C be a subgroup of G containing A, and let B be any semigroup from G. Following B. I. Plotkin (^{4}), we shall call a semigroup A **semi-Dedekind** if the equality

$$\{A, B\} \cap C = \{A, B \cap C\}$$

holds.

Lemma 6. If I is an isolated subgroup in a torsion-free group G, distinct from its normalizer $N(I)$, it contains a semi-Dedekind subsemigroup A with identity, then the normalizer $N(I)$ is contained in $N(A)$, the normalizer of A.

Proof. If $x \in N(I) \setminus I$, then by Lemma 5.2 of (4), $x \in N(A)$. Now let $x \in I,\ g \in N(I) \setminus I$; with the aid of Lemma 5.2 of (4) we obtain

$$A = x^{-1}g^{-1}Agx = x^{-1}Ax,$$

i.e. in this case also $x \in N(A)$. Thus $N(I) \subset N(A)$.

Lemma 7. Let, under a PS-isomorphism φ, the image I^φ of an isolated normal divisor I of a group G be an invariant subgroup in G^φ. Then the image A^φ of an invariant subsemigroup with identity A, containing I, is an invariant subsemigroup of G^φ.

Proof. A is invariant in G and, by Lemma 5.1 of (4), is a half-Dedekind subsemigroup in G. But then A^φ is half-Dedekind in G^φ and, by Lemma 6, A^φ is invariant in G^φ.

Theorem 3. Let G and G^φ be PS-isomorphic groups; let the group G be ordered, and let Γ be the subsemigroup of positive elements of the group G. Then the group G^φ can be ordered, and as the subsemigroup of positive elements in G^φ one may take Γ^φ.

Proof. First we prove the theorem in the case where G has a finite number of generators. From Lemma 1 of (1) it is known that Γ^φ in G^φ is a linear subsemigroup with identity and without inverse elements. Thus it remains only to prove the invariance of Γ^φ in G^φ. Let H be a maximal convex subgroup in G; it is invariant and isolated in G. The group G is ordered and therefore is an R-group. By Theorem 4, H^φ is an invariant subgroup in G^φ. The PS-isomorphism of the groups G and G^φ induces a PS-isomorphism of the factor groups G/H and G^φ/H^φ (5). G/H is abelian and, by (2), G^φ/H^φ is also abelian. The subsemigroup $(\Gamma H)^\varphi = \Gamma^\varphi H^\varphi$ is linear in G, and therefore

$$\Gamma^\varphi H^\varphi = \Gamma^\varphi \cup H^\varphi.$$

It contains the commutant, since $H^\varphi \subset \Gamma^\varphi H^\varphi$ and G^φ/H^φ is abelian. Therefore the subsemigroup $\Gamma^\varphi \cup H^\varphi$ is invariant in G^φ. By Lemma 9, the subsemigroup $(\Gamma \cap H)^\varphi = \Gamma^\varphi \cap H^\varphi$ is invariant in G^φ. From the invariance of the subsemigroups $\Gamma^\varphi \cup H^\varphi$, $\Gamma^\varphi \cap H^\varphi$, and the invariance of the subgroup H^φ in G^φ, it follows that Γ^φ is invariant in G^φ. The theorem is extended in an obvious manner to any ordered group G.

Lemma 8. Let φ be a proper PS-isomorphism. Let the series

$$E = H_0 \subset H_1 \subset \cdots \subset H_\alpha \subset \cdots \subset G$$

and

$$E^\varphi = H_0^\varphi \subset H_1^\varphi \subset \cdots \subset H_\alpha^\varphi \subset \cdots \subset G^\varphi$$

be ascending normal series in the R-groups G and G^{φ} such that the factors H_{i+1}/H_i, $H_{i+1}^{\varphi}/H_i^{\varphi}$ $(i = 1, 2, \dots, \alpha, \dots)$ are locally nilpotent torsion-free groups. Let φ be an isomorphism (respectively an anti-isomorphism) of the nonabelian subgroups H_1 and H_1^{φ}. Then φ is an isomorphism (respectively an anti-isomorphism) of the groups G and G^{φ}. If H_1 is abelian, then φ is an isomorphism or an anti-isomorphism of the groups G and G^{φ}.

Theorem 4. A PS-isomorphism of two ordered groups G and G^{φ} is always proper.

Theorem 5. Let the groups G and G^{φ} be PS-isomorphic and let the group G be ordered. Then the groups G and G^{φ} are isomorphic, and the PS-isomorphism φ is a consequence of their group isomorphism or anti-isomorphism.

Proof. It is evidently sufficient to prove the theorem for an ordered group G with a finite number of generators. By Theorem 3 we order the group G^{φ}, taking as the subsemigroup of positive elements the subsemigroup Γ^{φ}. By Theorem 4, the PS-isomorphism φ is proper. Since G has a finite number of generators, all convex subgroups in it are invariant. Let $H \subset F$ be two adjacent convex subgroups in G; then, by Theorem 2, φ is either an isomorphism or an anti-isomorphism

groups H and H^{φ}. The set of all convex subgroups containing H is an ascending normal series in G with torsion-free abelian factors. Accordingly, the set of images of the terms of this series is an ascending normal series in G^{φ} (Theorem 1). Under a PS-isomorphism, Γ-convex subgroups are mapped onto Γ^{φ}-convex ones (^{1}). If Γ is an invariant subsemigroup of G with identity and without inverse elements, then the definitions of a Γ-convex and of a convex subgroup in the group G coincide, if as the subsemigroup of positive elements of the group G one takes the subsemigroup Γ (^{6}). Therefore, if A and B are adjacent convex subgroups in G, then A^{φ} and B^{φ} are adjacent convex subgroups in G^{φ}. Consequently, all factors of the series in the group G^{φ} are torsion-free abelian. Hence, by Lemma 8, it follows that φ is an isomorphism or an anti-isomorphism of the groups G and G^{φ}.

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Note: Figure translations are in progress. See original paper for figures.