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Abstract

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MATHEMATICS

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ON THE THEORY OF LOCALLY STABLE GROUPS OF AUTOMORPHISMS

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1. As is known, there are deep connections between the external and internal properties of groups of automorphisms. The present paper is devoted to conditions under which an external property—the local stability of a group of automorphisms—entails its internal local nilpotency. This question has already been considered by a number of authors.

The notion of a locally stable group of automorphisms is the result of transferring the notion of a locally nilpotent group to the case of a pair (G, Φ), where Φ is a group of automorphisms of the group G. We give the definitions used in the paper (see also ([1])).

A set (group) of automorphisms Σ of a group G is called **stable** (relative to G) if in G there is an ascending normal Σ-admissible series in whose factors Σ induces the identity automorphisms (identities). Σ is called **finitely stable** if there is a corresponding finite series. A finite set Σ is called **locally stable** if in G there is a local system of Σ-admissible subgroups, in each of which Σ acts as a stable set. Finally, the group Φ is called **locally stable** if every finite subset of Φ is locally stable. Locally finitely stable groups of automorphisms are defined in the corresponding way.

2. **Theorem 1*.** *A finitely stable group of automorphisms of a free group is nilpotent.*

Proof. 1) We first note the following (known) fact. Let G be a group, H its normal divisor, and let σ be an automorphism of the group G inducing the identity on H and on G/H. Then for any element $g \in G$ the element

$$[g, \sigma] = g^{-1} \cdot \sigma(g)$$

is contained in the center of H. Indeed, if $h \in H$, then

$$\begin{aligned} h^{-1}[g,\sigma]h &= h^{-1} \cdot g^{-1} \cdot \sigma(g) \cdot h = h^{-1} \cdot g^{-1} \cdot \sigma(gh) \\ &= h^{-1} \cdot g^{-1} \cdot \sigma(ghg^{-1} \cdot g) = h^{-1} \cdot g^{-1} \cdot ghg^{-1} \cdot \sigma(g) = g^{-1} \cdot \sigma(g) = [g,\sigma]. \end{aligned}$$

From the computations just given it is also easy to obtain the following assertion:

If in a group G there is an ascending invariant Φ-stable series, then the commutant $[G\Phi]$ belongs to the centralizer of this series and, in particular, itself possesses an ascending central series.

* 2) Let G, H, and σ be as before, and suppose, in addition, that φ is an automorphism of the group G inducing the identity in G/H. Then for any $g \in G$ the relation

$$[g, [\sigma, \varphi]] = [[g, \sigma], \varphi^{-1}]$$

holds.

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* After the paper had already been sent to press, we became aware of a paper by P. Hall ([7]), in which a theorem coinciding with our Theorem 1 is proved. In this connection Theorem 1 should be regarded as a new proof of Hall' s theorem. Unlike Hall, we give a proof without passing to the holomorph.

For the proof, denote $[g, \sigma] = a,\ [g, \varphi] = h$. We have

$$\begin{aligned}[g, [\sigma, \varphi]] &= g^{-1} \cdot \sigma^{-1}\varphi^{-1}\sigma\varphi(g) = g^{-1} \cdot \sigma^{-1}\varphi^{-1}\sigma(gh) \\ &= g^{-1} \cdot \sigma^{-1}\varphi^{-1}(ga \cdot h) = g^{-1} \cdot \sigma^{-1}\varphi^{-1}(gha) \\ &= g^{-1} \cdot \sigma^{-1}(g \cdot \varphi^{-1}(a)) = g^{-1} \cdot g \cdot a^{-1} \cdot \varphi^{-1}(a) = a^{-1} \cdot \varphi^{-1}(a) = [[g, \sigma], \varphi^{-1}],\end{aligned}$$

as was required. The relation obtained improves the formula of the lemma from ([2]).

3) Let, further, G be a group, H its normal divisor, and Φ a group of automorphisms of the group G such that H is Φ-admissible and in G/H the group Φ induces the identity. Let, moreover, Σ be a normal divisor in Φ inducing the identity in H. Then the equality holds (for $g \in G$)

$$[g, [\Sigma, \Phi]] = [[g, \Sigma], \Phi].$$

Indeed, since $[\Sigma, \Phi] \subset \Sigma$, and Σ induces the identity in G/H and in H, every element of the left-hand side has the form $[g, \psi]$, where ψ is an arbitrary element of $[\Sigma, \Phi]$. The element ψ has the form

$$\psi = \prod_i [\sigma_{k_i}, \varphi_{j_i}]^{\varepsilon_i},$$

where $\sigma_{k_i} \in \Sigma,\ \varphi_{j_i} \in \Phi$, and $\varepsilon_i = \pm 1$. We have

$$[g, \psi] = \left[g, \prod_i [\sigma_{k_i}, \varphi_{j_i}]^{\varepsilon_i}\right] = \prod_i [g, [\sigma_{k_i}, \varphi_{j_i}]]^{\varepsilon_i} = \prod_i [[g, \sigma_{k_i}], \varphi_{j_i}^{-1}]^{\varepsilon_i} \in [[g, \Sigma], \Phi],$$

which proves the inclusion in one direction. The second inclusion is checked analogously.

Let us now put $\Sigma_i = [\Sigma, \Phi(i)]$. The subgroup Σ_i is a normal divisor in Φ and, just like Σ, induces the identity in H and G/H. Consequently,

$$[g, [\Sigma_i, \Phi]] = [[g, \Sigma_i], \Phi].$$

Hence, by induction, we obtain the relation

$$[g, [\Sigma, \Phi(n)]] = [[g, \Sigma], \Phi(n)] \qquad (n = 0, 1, 2, \ldots).$$

4) We pass to the proof of the theorem. Let

$$E = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

be a finite normal Φ-stable series in the group G. We shall prove the nilpotency of the group Φ by induction on the length of such a series. For $n = 1, 2$ the assertion is obvious. Suppose that it is true for $k \leq n-1$. Let Σ be the Φ-centralizer of the subgroup G_{n-1}. Then Σ is abelian and Φ/Σ, by the induction hypothesis, is nilpotent. From the preceding it follows that $[\Sigma, \Phi(n-1)]$ is the identity in Φ, i.e. Φ is a nilpotent group.

Recall that in the work of L. A. Kaluzhnin ([4]) an analogous theorem was proved for the case of an invariant stable series.

3. Theorem 2. *Let Φ be a locally stable group of automorphisms of a group G. Suppose that in the radical of the group G the periodic part is finite. Then the group Φ is locally nilpotent if and only if it has a local system consisting of subgroups of finite rank.*

The work ([5]) contains an example showing that local nilpotency of the group Φ cannot be derived from local stability of this group alone. On the other hand, it is obvious that every locally nilpotent group possesses a local system consisting of subgroups of finite rank.

We give auxiliary propositions.

Lemma 1. *Let Φ be a locally stable group of automorphisms of a group G. Let the periodic part P of the radical of the group G be finite and have order m, and let Φ be a group of finite rank r. Then the set of all elements of finite order in Φ is a subgroup of Φ, coinciding with the Φ-centralizer of the factor group G/P. This subgroup is finite and has order not exceeding the number $m!\,m^r$.*

The proof of this lemma is obtained from the following considerations. In ([5]) it is shown that every element of finite order from a locally stable group of automorphisms of some group G induces the identity automorphism in the factor group $G/P(R(G))$, where $P(R(G))$ is the periodic part in $R(G)$. Denoting by

Ψ the Φ-centralizer of the factor group G/P, we see that all elements of finite order from Φ belong to Ψ. The order of Ψ is estimated as follows.

Let Σ be the Ψ-centralizer of the subgroup P. Then the order of Ψ/Σ does not exceed $m!$. It is easy to see that Σ is a periodic group, and the orders of all elements of Σ are bounded by the number m. Since Σ is an abelian group of rank not exceeding r, it is clear that the order of Σ does not exceed m^r.

Lemma 2. *Let Φ be a stable group of automorphisms of the group G. Then, if Φ has a finite number of generators and finite rank, and in G the periodic part of the radical is finite, then Φ is nilpotent.*

Proof. Let r be the rank of Φ, m the order of the periodic part of the radical $R(G)$, and

$$E = G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset G_{\alpha+1} \subset \cdots \subset G_\gamma = G$$

a Φ-stable series of the group G. From Lemma 2 it follows that, both in the group Φ itself and in every group of automorphisms induced by the group Φ in the groups G_α, the periodic parts are finite and their orders do not exceed $m!\,m^r$. We shall prove the nilpotency of the group Φ by induction on the length of the Φ-stable series. For $\gamma = 1$ the assertion is obvious; suppose that it is true for all $\alpha < \gamma$. Let there exist $\beta = \gamma - 1$. Denote by $\mathfrak{Z}$ the Φ-centralizer of G_β. Then $\Phi/\mathfrak{Z}$ is nilpotent by the induction hypothesis, and $\mathfrak{Z}$ is abelian. We shall show that every element $\varphi \in \Phi$ induces a nil-automorphism in $\mathfrak{Z}$. Let σ be an arbitrary element in $\mathfrak{Z}$, and let g be some fixed element in G. The mapping $\sigma \to [g, \sigma]$ is a homomorphism of the group $\mathfrak{Z}$ onto a central subgroup A of G_β. A has finite rank, not exceeding r, and since, according to ([6]), A lies in the radical of the group G, the periodic part of A has order not exceeding m. From the relation

$$[[g, \sigma], \varphi^{-1}] = [g, [\sigma, \varphi]]$$

it follows that A is Φ-admissible. It is now clear that, if $s = r + m$, then $[A, \Phi(s)] = E$, and we emphasize that s does not depend on A, i.e. on the choice of the element g. Thus, from item 3) of the proof of Theorem 1 it follows that the automorphism $[\sigma, \varphi(s)]$ acts identically on every element $g \in G$. Hence, by the finiteness of the number of generators, it follows that Φ is a nilpotent group ([3]).

It remains to consider the case when γ is a limit ordinal. For this it is enough to refer to the following proposition.

Lemma 3. *Let the group G have a local system of Φ-admissible subgroups G_α, in each of which Φ induces a nilpotent group of rank not exceeding r, whose periodic part is finite and whose order does not exceed the number k. Then the whole group Φ is nilpotent.*

Proof. Let $\mathfrak{Z}_\alpha$ be the Φ-centralizer of the subgroup G_α. The intersection of all $\mathfrak{Z}_\alpha$ coincides with the identity of the group Φ; therefore Φ is a subdirect product of the groups $\Gamma_\alpha \simeq \Phi/\mathfrak{Z}_\alpha$. All Γ_α have nilpotency class not exceeding $r+k$. But a subdirect product of nilpotent groups whose nilpotency classes are bounded in the aggregate is again a nilpotent group. Thus Φ is a nilpotent group. In particular, Lemma 2 is also proved.

We pass to the proof of the theorem. Let G and Φ satisfy the hypotheses of the theorem. Without loss of generality one may assume that Φ has a finite number of generators and, consequently, finite rank.

Let $\{G_\alpha\}$ be a local system of Φ-admissible subgroups, in each of which Φ acts as a stable group. All commutants $[G_\alpha, \Phi]$ belong to the radical of the group G and, consequently, have periodic parts whose orders do not exceed the order of the periodic part of the radical. Let $\mathfrak{Z}_\alpha$ be the Φ-centralizer of G_α. Since in Lemmas 1 and 2 the radical can be replaced by the commutant $[G, \Phi]$, it now follows from these lemmas that all the groups $\Phi/\mathfrak{Z}_\alpha$ are nilpotent and that the orders of their periodic parts are bounded by some number k. Since the group Φ itself has finite rank, by Lemma 3 it is nilpotent.

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REFERENCES

[1] B. I. Plotkin, UMN, **13**, 4, 82 (1958).
[2] V. G. Vilyatser, DAN, **131**, No. 4 (1960).
[3] B. I. Plotkin, UMN, **9**, 3, 181 (1954).
[4] L. Kalujnin, Ber. Math. Tagung, Berlin, 164 (1953).
[5] V. G. Vilyatser, Uchen. zap. Ural. gos. univ. im. A. M. Gor' kogo, issue 23 (1959).
[6] B. I. Plotkin, DAN, **130**, No. 5 (1960).
[7] F. Holl, Illinois J. Math., **2**, No. 4 B (1958).

Note: Figure translations are in progress. See original paper for figures.