ON THE EXISTENCE OF APERIODIC FLOWS WITH A FREE BOUNDARY

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Abstract

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HYDROMECHANICS

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ON THE EXISTENCE OF APERIODIC FLOWS WITH A FREE BOUNDARY

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Many problems in the theory of wave motions of a liquid lead to the study of aperiodic flows whose free boundary has horizontal asymptotes. These include problems on the motion of a body beneath a free surface, on the flow of a liquid over an uneven bottom, and the problem of a solitary wave. Here the existence of a solution to the problem of the motion of a flow over an uneven bottom will be proved in the case of large Froude numbers. A special case of this problem, when the bottom of the flow descends in the direction of the current, was considered by R. Gerber ([1]). In addition, we shall show that in a liquid of infinite depth a solitary wave cannot exist. The existence of solitary waves in a liquid of finite depth, as is known, was first proved by M. A. Lavrent' ev.

The motion is assumed to be plane, steady, and irrotational; the liquid is ideal and incompressible.

Fig. 1 Fig. 2

Fig. 1 **Fig. 2**

1. Let a flow of liquid with free boundary Γ flow over an uneven bottom D (see Fig. 1). The bottom is horizontal everywhere except for a segment AB of finite length L. We shall specify the curve D by the parametric equation

$$\alpha = \alpha[l],$$

where α is the angle formed by the tangent to D with the horizontal; l is the arc length measured from the point C, the midpoint of the arc AB. We have

$$\alpha[l] = 0 \quad \text{for } |l| > L/2.$$

The function $\alpha[l]$ is assumed piecewise smooth.

Problem 1. Determine the liquid flow and the curve Γ, if the following are given: the function $\alpha[l]$; the discharge Q; L, the length of the arc AB; and the mean velocity

$$c = \frac{1}{L}\int_{-L/2}^{+L/2} |\bar{v}|\, dl.$$

To derive the basic equations, map the flow region onto the infinite strip $0 \leq \psi \leq h$ of the plane $w = \varphi + i\psi$. Under this mapping the free boundary Γ becomes the straight line $\psi = h$, the bottom D becomes the straight line $\psi = 0$, and the arc AB becomes the segment $0 \leq \varphi \leq 1$. We express the complex velocity $\bar{v}$ through the function

$$\omega(w) = c\Phi(\varphi;\psi) + i\Phi(\varphi;\psi),$$

putting

$$\bar{v} = ce^{-\omega(w)},$$

Using the conditions of constancy of the pressure along the free boundary and of flow past the bottom, we arrive at a system of equations, in analogy with how this is done in paper ([1]):

$$\Phi(\varphi) = -\frac{1}{2h}\int_{-\infty}^{\infty} \frac{c\Phi(\varphi) - c\Phi(u)}{\operatorname{sh}\frac{\pi}{2h}(u-\varphi)}\, du + \frac{1}{2h}\int_{-\infty}^{+\infty} \frac{\alpha[l(u)]}{\operatorname{ch}\frac{\pi}{2h}(u-\varphi)}\, du,$$

$$\frac{dc\Phi}{d\varphi} = \frac{gL}{c^2} e^{3c\Phi(\varphi)} \sin\Phi(\varphi), \qquad \frac{dl}{d\varphi} = Le^{c\Phi^*(\varphi)}.$$

Here the functions $\Phi(\varphi) = \overline{\Phi}(\varphi;h)$, $c\Phi(\varphi) = c\overline{\Phi}(\varphi;h)$, $l(\varphi)$ are unknown, while the function $c\Phi^*(\varphi) = c\overline{\Phi}(\varphi;0)$ is determined through $\Phi(\varphi)$ and $\alpha[l(\varphi)]$ as the limiting value of the conjugate $\overline{\Phi}(\varphi;\psi)$. We reduce this system to the operator equation

$$x = Ax, \tag{1}$$

where A is a completely continuous operator in a certain Banach space. If the Froude number is sufficiently large, then an a priori estimate for the solution of equation (1) can be obtained, after which the existence of a solution is a consequence of Schauder' s principle.

Theorem 1. *If* $\max|\alpha| < \pi/2$, *then one can indicate a positive number* ε *such that, for* $gL/c^2 \leqslant \varepsilon_h$, *a solution of problem 1 exists.*

This solution has the following properties:

1) $|\Phi[l]| \leqslant M_1 e^{-k|l|}$, $k > 0$, M_1 does not depend on l;

2) $\max_l |\Phi[l]| \leqslant M_2 h/L$, where $h = Q/c$ is the mean depth and M_2 does not depend on the ratio h/L. Here * $\Phi = \Phi[l]$ is the parametric equation of the free boundary.

The following problem concerns flows symmetric with respect to the vertical axis.

Problem 2. Determine the fluid flow and the curve Γ so that the conditions

$$\Phi[l] = -\Phi[-l]; \qquad \Phi[l] \geqslant 0 \quad \text{for } l \leqslant 0,$$

are satisfied, if the following are given: the function $\alpha[l]$; the discharge Q; L, the length of the arc AB; and c_0, the velocity at the point C', the vertex of the free boundary.

For problem 2 a more precise result is valid, obtained by the same methods as for problem 1.

Theorem 2. *Suppose the following conditions are fulfilled:*

1) $gQ/c_0^3 < 1$;
2) $gQ/c_0^3 + \max|\alpha| < \pi/2$;
3) $\alpha[-l] = -\alpha[l]$, $\alpha[l] \geqslant 0$ for $l \leqslant 0$.

Then a solution of problem 2 exists.

2. Suppose that on the surface of a fluid a solitary wave propagates with constant velocity c without change of form. At infinity the fluid is at rest. We shall assume that the solitary wave (see Fig. 2) has the following properties: 1) the free boundary has one vertex; 2) the solitary wave is symmetric with respect to the vertical axis passing through the vertex; 3) the height of the solitary wave is finite.

Theorem 3. *Solitary waves in a fluid of infinite depth do not exist.*

Let us note that in a fluid of infinite depth there also cannot exist periodic waves of arbitrarily great length, if their propagation velocity is bounded ([3]).

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* Φ is the angle formed by the tangent to the profile of the free boundary with the horizontal.

For a liquid of finite depth h (h is the depth of the liquid at infinity) the following result is valid:

Theorem 4. *If the Froude number is less than one, i.e. $gh/c^2 > 1$, then solitary waves do not exist.*

A consequence of the last theorem is the fact that a solitary wave possessing properties 1), 2), 3) must necessarily be a wave of elevation.

The proof of Theorems 3 and 4 is based on a method for estimating the bounds of the positive spectrum, widely used in the theory of positive operators ([2]).

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Note: Figure translations are in progress. See original paper for figures.