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Abstract

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EXISTENCE THEOREMS FOR NONLINEAR MIXED PROBLEMS

(Presented by Academician S. L. Sobolev on 9 VI 1960)

The establishment of conditions for the existence and uniqueness of solutions of nonlinear mixed problems is the subject, for example, of the works ($^{1-7}$). Analogous questions for the Cauchy problem have been solved in the works (^{8,9}). In the present paper, the approach proposed in (^{11}) for justifying linearization in the study of stability is used to obtain existence theorems in general, uniqueness, continuous dependence, and differentiability with respect to a parameter in the following nonlinear problems:

Problem 1. In the strip $\Pi_T = \{0 \leq x \leq 1,\ 0 \leq t \leq T\}$, find an n_0-times ($n_0 = \max(2, n)$) continuously differentiable solution $u(x,t)$ of the equation

$$u_{xx} - u_{tt} = -f(x, t, u, u_x, u_t, \mu), \tag{1}$$

satisfying the boundary conditions ($i = 1, 2;\ 0 \leq s \leq n-1$)

$$\sum_{j=0,1} \sum_{k+l=n} \alpha_{k,l}^{(i,j)}(t,\mu) \left. \frac{\partial^{(n)} u}{\partial x^k \partial t^l} \right|_{x=j} = \varphi_i(t, \dots, D^{(s)} u|_{x=0}, D^{(s)} u|_{x=1}, \dots, \mu) \tag{2}$$

and the initial conditions

$$u(x,0) = \psi_0(x,\mu), \qquad u_t(x,0) = \psi_1(x,\mu). \tag{3}$$

Problem 2. In the strip Π_T, find an n_0-times continuously differentiable* solution of the equation

$$u_t - u_{xx} = f(x, t, u, u_x, \mu), \tag{4}$$

satisfying the boundary conditions ($i = 1, 2;\ 0 \leq p + 2q \leq n-1$)

$$\sum_{j=0,1}\sum_{k+2l=n}\alpha_{k,l}^{(i,j)}(t,\mu)\left.\frac{\partial^{(n)}u}{\partial x^k\partial t^l}\right|_{x=j}=\varphi_i\left(t,\dots,\left.\frac{\partial^{(s)}u}{\partial x^p\partial t^q}\right|_{x=0},\left.\frac{\partial^{(s)}u}{\partial x^p\partial t^q}\right|_{x=1},\dots,\mu\right) \tag{5}$$

and the initial condition

$$u(x,0)=\psi_0(x,\mu). \tag{6}$$

1. Suppose that for Problems 1 and 2 the following assumptions are fulfilled:

1) For every $0<R<+\infty$, when $(x,t)\in\Pi_T$, $-R<v_1,v_2,v_3<+R$, all partial derivatives of the function $f(x,t,\ldots)$ up to order n_0-1 inclusive exist, are continuous, and satisfy a Lipschitz condition with constant $C_1(R,T)$, depending only on R and T.

*

Speaking of Problem 1 or 2, by an m-times continuously differentiable function $g(x,t,v_1,\dots,v_k,\mu)$ we shall mean a function for which all derivatives of the form $\partial^{(s)}g/\partial x^\alpha\partial t^\beta\partial v_1^{\gamma_1}\cdots\partial v_k^{\gamma_k}$ exist and are continuous, where $\alpha+\beta+\gamma_1+\dots+\gamma_k=s\le m$, or $\alpha+2\beta+\gamma_1+\dots+\gamma_k=s\le m$, respectively.

2) For any $0<R<+\infty$, when $(x,t)\in\Pi_T$, $-R<v_1,v_2,\ldots<+R$, the functions $\varphi_i(t,v_1,\ldots)$, and for problem 1 also all derivatives $\partial\varphi_i/\partial t$, $\partial\varphi_i/\partial v_k$, are continuous and satisfy a Lipschitz condition with respect to the variables $v_1,v_2,\ldots$, with constant $C_2(R,T)$.

3) For $0\le t\le T$, the coefficients $\alpha_{k,l}^{(i,j)}(t,\mu)$ are continuous in t, and in the case of problem 1

$$(\gamma_1^0-\delta_1^1)(\gamma_2^1+\delta_2^1)-(\gamma_2^0-\delta_2^1)(\gamma_1^1+\delta_1^1)\neq 0,$$

while in the case of problem 2

$$\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21}\neq 0,$$

where

$$\gamma_i^j=\sum_{2p\le n}\alpha_{2p,n-2p}^{(i,j)},\qquad \delta_i^j=\sum_{2p+1\le n}\alpha_{2p+1,n-2p-1}^{(i,j)},\qquad \gamma_{ij}=\sum_{k+2l=n}\alpha_{k,l}^{(i,j)}.$$

4) The functions $\psi_0(x,\mu)$ and $\psi_1(x,\mu)$ have, respectively, n_0 and n_0-1 continuous derivatives with respect to x on the segment $[0,1]$.

5) The compatibility conditions are satisfied.

Then the following theorems hold for problems 1 and 2.*

Theorem 1. Either in the strip Π_T there exists a unique n_0-times continuously differentiable solution, or there is such a $T_i^*,\ 0 < T_i^* \leq T$, that in $\Pi_{T_i^*-\varepsilon}$, for any $\varepsilon > 0$, there exist unique n_0-times continuously differentiable solutions $u_i(x,t)$ $(i = 1, 2)$ of problems 1 and 2, respectively, and

$$\max_{\substack{0\leq x\leq 1\\ 0\leq t\leq T_i^*-\varepsilon}} |u_i| + \max_{\substack{0\leq x\leq 1\\ 0\leq t\leq T_i^*-\varepsilon}} \left|\frac{\partial u_i}{\partial x}\right| + (2-i) \max_{\substack{0\leq x\leq 1\\ 0\leq t\leq T_i^*-\varepsilon}} \left|\frac{\partial u_i}{\partial t}\right| +$$

$$+ \sum_{j=0,1} \sum_{2\leq k+l\leq n-1} \max_{0\leq t\leq T_i^*-\varepsilon} \left|\frac{\partial^{(k+l)} u_i(j,t)}{\partial x^k \partial t^l}\right| \to \infty \quad \text{as } \varepsilon \to 0.$$

Theorem 2. If conditions 1)-5) are satisfied for any $\mu \in (\underline{\mu}, \bar{\mu})$, and the functions occurring in them and their derivatives are continuous in μ, then the solutions of problems 1 and 2, together with all derivatives with respect to x and t up to order n_0 inclusive, are continuous in μ in the same interval.

Theorem 3. If conditions 1)-5) are satisfied for any $\mu \in (\underline{\mu}, \bar{\mu})$, and the functions occurring in them and their derivatives are continuously differentiable with respect to μ, then the solutions of problems 1 and 2, together with all derivatives up to order n_0 inclusive, are continuously differentiable with respect to μ.

The proof of Theorems 1, 2, and 3 is based on reducing problems 1 and 2 to an equivalent integral form, with subsequent use of Theorems 4, 5, and 6 formulated below. To reduce problems 1 and 2 to integral form, we first consider the linear problems obtained by replacing the right-hand sides of equations (1), (2) and (4), (5) by certain functions $f(x,t)$ and $\varphi_i(t)$ independent of the variables x and t, whose solution is written in the form

$$u(x,t) = u_0(x,t) + \int_0^t \int_0^1 G(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau +$$

$$+ \int_0^t G_1(x, t-\tau) \left[\varphi_1(\tau) + L_1 f\right] d\tau + \int_0^t G_2(x, t-\tau) \left[\varphi_2(\tau) + L_2 f\right] d\tau, \qquad (7)$$

where $L_1 f$ and $L_2 f$ are linear combinations of derivatives of the function $f(x,t)$ up to order $n_0 - 1$ inclusive, evaluated at $x = 0,\ x = 1$, respectively.

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[*] Theorems 1, 2, and 3 remain valid also in the case when terms of the form

$$\nu \chi_i(t, \dots, D^{(s)} u|_{x=0}, D^{(s)} u|_{x=1}, \dots, \mu), \quad 0 \leq s \leq n,$$

are added to the right-hand sides of the boundary conditions (2), (5), where ν is sufficiently small and the functions χ_i satisfy the same conditions as the functions φ_i. Moreover, we note that the case when in equations (1) and (4), before the second derivative with respect to x, there stands an n_0-times continuously differentiable function $a(x,t) > a_0 > 0$ in Π_T, entails no changes in Theorems 1, 2, and 3.

but $u_0(x,t)$ is the solution of the corresponding homogeneous problem, satisfying the initial conditions (3) or (6). Substituting in (7), instead of the functions $f(x,t)$, $\varphi_i(t)$, the functions $f(x,t,...)$, $\varphi_i(t,...)$, respectively, we obtain the required integral equation. Differentiating the obtained equation with respect to x and t, we arrive at a system of integral equations for the function $u(x,t)$ and all its partial derivatives up to order n_0, inclusive.

We note that the reduction of problem 1 to integral form, with the subsequent application of Theorems 4, 5, and 6, can be obtained by the method of continuation of the initial conditions proposed for nonlinear mixed problems for equations of hyperbolic type in the papers (^{12}).

Considering the collection of derivatives $u_{\alpha\beta}$ ($0 \le \alpha + \beta \le n_0$ for problem 1 and $0 \le \alpha + 2\beta \le n_0$ for problem 2) as components of the vector $\mathbf{u}(x,t)$, we write the resulting system of integral equations in the form

$$\mathbf{u} = \Omega \mathbf{u}, \tag{8}$$

where the operator Ω is dynamic in the sense of the definition given below. If the norm of the vector $\mathbf{u}$ is defined as

$$\|\mathbf{u}\|_T = \max_{0 \le t \le T} \left\{ \sum_{\alpha,\beta} \max_{0 \le x \le 1} |u_{\alpha\beta}(x,t)| \right\},$$

then for equation (8), under the assumptions of Theorems 1, 2, and 3, respectively the assumptions of Theorems 4, 5, and 6 are fulfilled. The assertions of Theorems 1, 2, and 3 follow from Theorems 4, 5, and 6.

2. Let H_T be the space of continuous functions of the real variable $0 \le \tau \le T < +\infty$ with values in the Banach space B. The value of a function $h \in H_T$ at $\tau = t$ will be denoted by h_t. In the space H_T we introduce the norm

$$\|h\|_T = \max_{0 \le t \le T} \|h_t\|.$$

An operator Ω mapping H_T into itself will be called dynamic if, for any $0 \le t \le T$, from

$$h_\tau = g_\tau \quad \text{for } \tau < t \text{ and } \tau \ge 0 \tag{9}$$

it follows that $(\Omega h)_t = (\Omega g)_t$.

Assume that for arbitrary functions h and g satisfying condition (9), and $t \leq \bar{t} \leq T$,

$$\|\Omega h - \Omega g\|_{\bar{t}} \leq \delta(t, \bar{t})\, C(R) \|h - g\|_{\bar{t}}, \tag{10}$$

where $\delta(t, \bar{t}) \to 0$ as $t - \bar{t} \to 0$, and $R = \max\{\|h\|_{\bar{t}}, \|g\|_{\bar{t}}\}$. Let Ω be an arbitrary dynamic operator defined in the space H_T and satisfying (10). Then the following theorems hold concerning the solutions of equation (8).

Theorem 4. *Equation (8) is either uniquely solvable in the space* H_T*, or there exists such a* $0 < t^* \leq T$ *that a solution* x *of equation (8) exists in the space* $H_{t^*-\varepsilon}$ *for every* $\varepsilon > 0$*, it is unique, and*

$$\|x\|_{t^*-\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

Theorem 5. *Let the dynamic operator* $\Omega(\mu)$ *be defined in the space* H_{T_1} *for all* $\mu \in (\underline{\mu}, \bar{\mu})$*, satisfy condition (10), and, for some* $\mu_0 \in (\underline{\mu}, \bar{\mu})$*,*

$$\|\Omega(\mu)h - \Omega(\mu_0)h\|_{T_1} \to 0 \quad \text{as} \quad \mu \to \mu_0$$

for an arbitrary function $h \in H_T$*. Suppose that for* $\mu = \mu_0$ *there exists a solution of equation (8) in the space* H_{T_1}*,* $0 < T_1 < T$*. Then there is a number* $\delta(\mu_0, T_1) > 0$ *such that, for every* μ *satisfying the inequality* $|\mu - \mu_0| < \delta(\mu_0, T_1)$, there exists a solution $x(\mu)$ of equation (8) in the space H_{T_1}, and

$$\|x(\mu) - x(\mu_0)\|_{T_1} \to 0 \quad \text{as } \mu \to \mu_0.$$

Theorem 6. Let the dynamical operator $\Omega(\mu)$ be defined in the space H_T for all $\mu \in (\underline{\mu}, \overline{\mu})$; let, for any $\mu_0 \in (\underline{\mu}, \overline{\mu})$ and $h_0 \in H_T$, there exist partial derivatives * Ω_μ, Ω_h continuous jointly in the variables μ_0, h_0. Then, if equation (8) has a solution $x(\mu)$ in the space H_T for every $\mu \in (\underline{\mu}, \overline{\mu})$, the function $x(\mu)$ is continuously differentiable with respect to μ on the interval $(\underline{\mu}, \overline{\mu})$.

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* According to the definition, a dynamic operator Ω defined on the space H_T is naturally also defined on any space $H_{T'}$, with $0 < T' < T$.

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∗ The derivatives are understood in the strong sense (^{10}).

Note: Figure translations are in progress. See original paper for figures.

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