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Abstract

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CONJUGATE MANIFOLDS AND THEIR APPLICATIONS

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1. As is known, the lines of three-dimensional space P_3 are mapped to the points of the hyperquadric Q_4^2 of five-dimensional space P_5 ($^{2-4}$). The hyperquadric establishes a polar correspondence of linear subspaces in P_5, and the sum of the dimensions of two polar-conjugate spaces is equal to 4. If one subspace lies on a tangent hyperplane of Q_4^2 and passes through the point of tangency, then its conjugate also satisfies this requirement. Such subspaces we shall call **conjugate manifolds**. This concept can also be established without the polar correspondence of Q_4^2, by finding the characteristics L_{4-m} of the tangent hyperplane along the given manifold L_m. The manifolds L_m and L_{4-m} are conjugate to one another. The common subspace of two conjugate manifolds L_m and L_{4-m} is called an **asymptotic manifold**. It is proved that only along asymptotic manifolds is the contact of the hyperplane and the hyperquadric of order not lower than the second at the point of tangency. These arguments, carried out for Q_4^2 in P_5, remain unchanged for an arbitrary hypersurface in a space of any dimension and make it possible to establish conjugate manifolds on a surface of any dimension in P_n. We shall verify this below on examples of congruences and complexes.

It is easy to prove that the asymptotic manifolds for Q_4^2 are either lines (generators), or 2-planes (generating planes). These manifolds possess the following properties: 1) through each point of Q_4^2 there pass ∞^2 generating lines and ∞^1 generating 2-planes; 2) through each generating line there pass two generating 2-planes; 3) each generating 2-plane is either the image of a point (the image of the bundle of lines with center at this point), or the image of a plane (the image of all lines lying in this plane) from P_3; thus, all generating 2-planes are divided into two series; 4) any two planes of one series have one common point, and any two planes from different series either have no common point, or have a common line; 5) the quadric Q_4^2 has no generating 3-plane.

2. In this note we shall apply the arguments set forth to the hyperquadric Q_4^2 in P_5. The work employs Cartan’ s method of exterior forms (^{1}), and

the Grassmann product is used extensively for finding characteristics.

The infinitesimal displacement of the tetrahedron $A_1A_2A_3A_4$ in projective space is determined by the equations

$$dA_i = \omega_i^k A_k, \qquad i, j, k = 1, 2, 3, 4, \tag{1}$$

where ω_i^k are linear differential forms connected by the structural equations of the space $D\omega_i^k = [\omega_i^j \omega_j^k]$ ([1]).

For brevity, let us denote the analytic lines by the letters

$$p_1 = (A_1A_2), \qquad p_2 = (A_3A_4), \qquad p_3 = (A_2A_3), \qquad p_4 = (A_1A_4),$$

$$p_5 = (A_1A_3), \qquad p_6 = (A_4A_2). \tag{2}$$

The tangent hyperplane of the quadric Q_4^2 at the point p_1 is determined by the Grassmann product of 5 points $(p_1p_3p_4p_5p_6)$. The asymptotic variety is determined from the condition $(d^2p_1p_3p_4p_5p_6p_1) = 0$, which leads to the equation

$$\omega_1^3\omega_2^4 - \omega_2^3\omega_1^4 = 0. \tag{3}$$

The left-hand side of this equation is a nondegenerate quadratic form (the rank is 4 ([5])).

When A_1A_2 in P_3 describes a ruled surface, its image in P_5 describes a curve on Q_4^2; moreover, if the ruled surface is developable, then its image in P_5 touches the generators of Q_4^2 (has an asymptotic direction). A line belonging to a generating plane of Q_4^2 is the image either of a cone or of a one-parameter family of lines lying in a plane in P_3.

The most general ruled surface L, or its image l on Q_4^2, can be defined by the equations

$$\omega_2^4 = \lambda\omega_1^3, \qquad \omega_1^4 = 0, \qquad \omega_2^3 = 0. \tag{4}$$

The line $(p_1, \lambda p_4 - p_3)$ is tangent to the line l, and the 3-plane $(p_1, \lambda p_4 + p_3, p_5, p_6)$ is its conjugate variety.

The characteristic of this 3-dimensional plane along (4) is a 2-dimensional plane that is conjugate to the osculating 2-plane of the curve l, i.e. both of them are images of the two series of rectilinear generators of the Lie quadric of the surface L.

3. Let us apply the preceding considerations to the theory of congruences. The image of a congruence K in P_5 is a two-dimensional surface k on Q_4^2. The tangent 2-plane of the surface k, depending on whether the congruence is hyperbolic or parabolic, intersects Q_4^2 in two intersecting (at the point of tangency) generator lines and in one generator line. Congruences are very easily classified by these properties. The most general hyperbolic (or elliptic) congruence k, referred to a tetrahedron of the first order, is defined by equations (2)

$$\begin{aligned} &\omega_1^4 = 0, \qquad \omega_2^1 = \beta\omega_1^3 + \gamma\omega_2^4, \qquad \omega_3^4 = \alpha\omega_1^3 - \beta\omega_2^4, \\ &\omega_2^3 = 0, \qquad \omega_2^1 = \gamma'\omega_1^3 + \beta'\omega_2^4, \qquad \omega_4^3 = -\beta'\omega_1^3 + \alpha'\omega_2^4. \end{aligned} \tag{5}$$

Substituting these values into (3), we obtain $\omega_1^3\omega_2^4 = 0$, i.e. the asymptotic ruled surfaces of the congruence coincide with the developable surfaces. An arbitrary ruled surface L and its image in the congruence (5) are defined by the equation $\omega_2^4 = \lambda\omega_1^3$. The three-dimensional plane $(p_1, \lambda p_4 + p_3, p_5, p_6)$ is the conjugate variety to the line l, which intersects the tangent plane $(p_1p_3p_4)$ of the congruence k along the line $(p_1, \lambda p_4 + p_3)$. This line touches the line $l' \subset k$, determined by the equation $\omega_2^4 = -\lambda\omega_1^3$. As is known, these two ruled surfaces are conjugate (in the sense of Sannia ([6–8])). It is proved that the pair of tangents to the conjugate lines l and l' harmonically separates the pair of asymptotic tangents. Here a more general theorem is obtained:

The cross ratio of four tangent planes to four ruled surfaces L_i of a congruence does not depend on the position of the common point of tangency on the ray of the congruence and is equal to the cross ratio of four tangent lines to the corresponding lines l_i in P_5.

Since there exists an unlimited number of ruled surfaces G_i of congruences possessing known geometric properties ([9]), by virtue of this theorem, for the corresponding lines g_i in P_5 we obtain an unlimited number of invariants. In particular, the Welsch invariant ([2]) can be defined in a new way: the square of the cross ratio of two asympto-

tic tangents and of two tangents to lines corresponding to asymptotic curves of different focal surfaces (from each focal surface one asymptotic is taken, no matter which) is the Welch invariant (see ([2]), p. 351).

The characteristic of the 3-dimensional plane $(p_1, \lambda p_4 + p_3, p_5, p_6)$ along the line $l(\omega_2^4 = \lambda\omega_1^3)$ is a 2-plane:

$$\begin{gathered} \{p_1(\lambda_1 + \lambda\lambda_2) - 2(\lambda p_4 + p_3), \\ 2\lambda p_5 - (\alpha - 2\lambda\beta - \gamma\lambda^2)p_1,\ 2\lambda p_6 + (\gamma' + 2\lambda\beta' - \alpha'\lambda^2)p_1\}, \\ d\ln\lambda + \omega_1^1 - \omega_2^2 - \omega_3^3 + \omega_4^4 = \lambda_1\omega_1^3 + \lambda_2\omega_2^4. \end{gathered} \tag{6}$$

The 2-plane (6) gives the second series of generators belonging to the Lie quadric of the surface L ([9–11]). The first series of generators of this quadric gives the osculating plane to the line l. The plane (6) and the analogous plane for the conjugate direction intersect the tangent plane $(p_1p_3p_4)$ of the congruence p_1, each in one point

$$p = (\lambda_1 + \lambda\lambda_2)p_1 - 2(\lambda p_4 + p_3), \quad p' = (\lambda_1 - \lambda\lambda_2)p_1 - 2(-\lambda p_4 + p_2). \tag{7}$$

Thus, with each ruled surface L of the congruence there is associated a definite invariant linear complex. The line pp' intersects Q_4^2 in two points

$$\lambda_1 p_1 - 2p_3, \quad \lambda_2 p_1 - 2p_4. \tag{8}$$

In P_3 the lines (8) are new transformations of the congruence (A_1A_2) in two conjugate directions.

The plane (6), with the 2-plane $(p_1p_5p_6)$ conjugate to the congruence (5), intersects in a line, and the latter has two common points with Q_4^2

$$2\lambda p_5 + (\gamma\lambda^2 + 2\beta\lambda - \alpha)p_1, \quad 2\lambda p_6 - (\alpha'\lambda^2 - 2\beta'\lambda - \gamma')p_1. \tag{9}$$

In P_3 the lines (9) give another transformation of the congruence (A_1A_2) in any direction $\omega_2^4 = \lambda\omega_1^3$.

The characteristic of the manifold $(p_1p_5p_6)$ conjugate (to the congruence) in the direction $\omega_2^4 = \lambda\omega_1^3$ is an invariant linear complex (a point in P_5)

$$\{\beta\gamma' + \alpha\beta' + \lambda(\gamma\gamma' - \alpha\alpha') + \lambda^2(\beta'\gamma + \beta\alpha')\}p_1 + (\alpha'\lambda^2 + \gamma')p_5 + (\gamma\lambda^2 + \alpha)p_6. \tag{10}$$

For developable surfaces ($\lambda = 0,\ \lambda = \infty$) (10) coincides with the known complexes ([12]), obtained by S. P. Finikov (([2]), p. 373) by analytic methods. For ruled surfaces corresponding to asymptotic lines of the focal surfaces ($\lambda^2 = -\alpha/\gamma$ or $\lambda^2 = -\gamma'/\alpha'$, ([2]), p. 351), (10) lies on Q_4^2 and is a new transformation of the congruence. The linear complex (10) does not depend on λ if and only if (A_1A_2) is a W-congruence. In this case (10) coincides with the osculating linear complex of the congruence (([2]), p. 375).

4. Conjugate manifolds in P_5 also have broad application to pairs of congruences. We give here several theorems for them.

Two congruences K and K' form a pair T ([3]) if and only if their tangent 2-planes in P_5 intersect in a line. The latter line intersects Q_4^2 in two points which are the images of the diagonals of the pair. Consequently, *the tangent planes of the auxiliary pair T also intersect along this line, i.e. all four tangent* *planes of the given pair T and of its auxiliary pair form a pencil of planes in P_5. The diagonals of the pair T form a new pair T if and only if the first pair is a Bianchi configuration* ([3],[11]).

If the tangent planes of two congruences K and K' intersect in only one point, then a new pair A_0 ([13]) is obtained, which is characterized by the fact that it contains at least one pair of ruled

scattering surfaces (see ([3]), pp. 305-324). If the pair T is composed of congruences W, then its diagonals form a pair A_0. Two congruences K and K' constitute a pair θ ([3],[14],[15]) if and only if the tangent plane of one congruence in P_5 has a common line with the conjugate 2-plane of the other congruence. This line has two points on Q_4^2, which are the images of the auxiliary opposite rays of the pair θ. These theorems are simple criteria for determining the corresponding configurations.

5. For hyperbolic congruences the rank of the quadratic form (3) is equal to 2 with respect to any coordinate frame (([5]), pp. 152-166). It is lowered only for parabolic congruences (the case of degeneration). It is proved that every ruled surface of a parabolic congruence is conjugate to a unique asymptotic (developable) ruled surface. This is an analogue of conjugacy on developable surfaces in P_3.

6. As is known, a line complex with respect to a tetrahedron of the first order is defined by the following equations (([3]), pp. 386-392)

$$\omega_1^4 = \omega_2^3, \qquad \varphi_i = a_{ij}\omega^j, \qquad a_{ij} = a_{ji}, \qquad i, j = 1, 2, 3$$

$$(\varphi_1 = [[unclear: term]] - \omega_2^1 - \omega_3^4, \qquad \varphi_2 = \omega_1^1 - \omega_2^2 - \omega_3^3 + \omega_4^4, \qquad \varphi_3 = \omega_1^2 + \omega_4^3,$$

$$\omega_1^3 = \omega^1, \quad \omega_2^3 = \omega^2, \quad \omega_2^4 = \omega^3)\,. \tag{11}$$

The characteristic of the tangent 3-plane of the complex (11) is a 2-plane only in four directions

$$2\omega^2\varphi_1 + \omega^3\varphi_2 = 0, \qquad 2\omega^2\varphi_3 + \omega^1\varphi_2 = 0, \tag{12}$$

while for the remaining directions it is a straight line in P_5. Each ruled surface L of the complex $\omega^3 = \lambda\omega^1$, $\omega^2 = \mu\omega^1$ is conjugate to a 3-plane, and the latter intersects the tangent 3-plane of the complex (11) in a 2-plane which is tangent to the congruence $\omega^3 = -\lambda\omega^1 + 2\mu\omega^2$. This congruence and the ruled surface L are conjugate to one another in the complex (11), and if the congruence is parabolic, then L is developable, and conversely. The tangent 3-plane of the complex (11) is conjugate to the line $(p_1,\ p_5 - p_6)$. This line has a characteristic only in the directions (12). The characteristic is determined by the point $ap_1 + p_5 - p_6$, where a is determined from $\varphi_2 + 2a\omega^2$ with the help of (11) and (12). Thus, these results make it possible to decompose the complex (11) into invariant ruled surfaces and congruences, and also to construct invariant linear complexes for (11). In this scheme the pair T of complexes ([3]) is characterized by the following criterion: two complexes constitute a pair T if and only if their tangent 3-planes in P_5 intersect in a 2-plane (or, what is the same, when their conjugate lines intersect).

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[1] S. P. Finikov, *Cartan' s Method of Exterior Forms*, 1948. [2] S. P. Finikov, *Theory of Congruences*, 1950. [3] S. P. Finikov, *Theory of Pairs of Congruences*, 1956. [4] S. P. Finikov, *Projective-Differential Geometry*, 1937. [5] A. G. Kurosh, *Higher Algebra*, 1946. [6] G. Sannia, *Ann. di Matem.* (III), 15, 143 (1908). [7] R. N. Shcherbakov, *Matem. sbornik*, 37 (79), No. 3 (1955). [8] R. N. Shcherbakov, *Matem. sbornik*, 46 (88), No. 2 (1958). [9] S. E. Karapetyan, *DAN*, 122, No. 3 (1958). [10] S. E. Karapetyan, *DAN*, 117, No. 2 (1957). [11] S. E. Karapetyan, *Scientific Reports of Higher Schools, Phys.-Math. Series*, No. 2 (1958). [12] S. E. Karapetyan, *Reports of the Academy of Sciences of the Armenian SSR*, 25, 3 (1957). [13] S. E. Karapetyan, *Izv. AS ArmSSR*, No. 4 (1959). [14] S. E. Karapetyan, *Collection of Scientific Works of the Armenian State Pedagogical Institute named after Kh. Abovyan*, 5 (1955). [15] S. E. Karapetyan, *Matem. sbornik*, 41 (83), 2 (1957).

Note: Figure translations are in progress. See original paper for figures.