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Abstract

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AN EXAMPLE OF A WEB NOT APPROXIMABLE BY RECTIFIABLE WEBS

(Presented by Academician A. N. Kolmogorov on 17 XI 1959)

In ([1,2]) we constructed an example of a function, defined on a grid, that is not nomographable but is approximable by nomographable functions, as well as an example of a function* that is not nomographable** in the unit square. The aim of the present note is to construct an example of a function that is no longer approximable by nomographable functions.

1. A **web** $S = \{A, B, C\}$ in a domain*** G will mean three families A, B, and C of curves in the domain G having the following properties: 1) through each point of the domain G there passes exactly one curve of each of the families A, B, and C; 2) any two curves belonging to two different families intersect in no more than one point; 3) for each pair of these families there exists a topological mapping of the domain G under which all curves of this pair of families go over into rectilinear segments.

By the **deviation** $d(S, S')$ of two webs $S = \{A, B, C\}$ and $S' = \{A', B', C'\}$ in the domain G we mean the number

$$d(S, S') = \sqrt{[d(A, A')]^2 + [d(B, B')]^2 + [d(C, C')]^2},$$

where $d(A, A')$ is the larger of the two numbers $\sup_{a \in A} (\inf_{a' \in A'} d(a, a'))$ and $\sup_{a' \in A'} (\inf_{a \in A} d(a, a'))$ ($d(a, a')$ is the deviation of two sets a and a', see ([3]), p. 166), and $d(B, B')$ and $d(C, C')$ are defined analogously. The deviation $d(S, S')$ satisfies all the axioms of a metric.

Let $z = f(x, y)$ be a function defined in the rectangle

$$R: \quad \underline{x} \le x \le \overline{x}, \quad \underline{y} \le y \le \overline{y}.$$

Then the two families of rectilinear segments $x = \text{const}$ and $y = \text{const}$ ($\underline{x} \le x \le \overline{x}$, $\underline{y} \le y \le \overline{y}$) and the family of level lines $z = \text{const}$ form a web in R. We shall call it the web corresponding to the function $z = f(x, y)$.

Lemma. *Let $f_n(x, y)$ be a sequence of functions converging uniformly in R to the function $f(x, y)$. Then the sequence of webs corresponding to the functions $z = f_n(x, y)$ converges to the web corresponding to the function $z = f(x, y)$.*

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Fig. 1

Figure 1: Fig. 1

A web $S = \{A, B, C\}$ in a domain G will be called **rectifiable** if there exists a topological mapping of the domain G under which all curves of **all three families** A, B, and C go over into rectilinear segments. The function $z = f(x, y)$ is nomographable in R if and only if

* In this note all functions of two variables are assumed to be strictly monotone in each of the variables when the other is held constant, and continuous in the domain under consideration.

** We are always speaking of functions that are nomographable or non-nomographable by means of continuous functions ([2]).

*** By a domain we always mean a planar set homeomorphic to a closed square.

case if the corresponding braid is rectifiable. Thus, in order to verify that some function is not nomographable, it suffices to show that at least some of the curves of the braid corresponding to it intersect one another in an order in which straight lines cannot intersect. Therefore an important problem is to find such properties of straight lines from which it follows that a certain order of the intersection points of several lines is possible for curves (elements of the families entering into the braid), but impossible for straight lines. The construction ([1,2]) of the above-mentioned example of a function that is not nomographable, but is approximable by nomographable functions, was based on one such property, concerning three quintuplets of straight lines in the plane. However, with the aid of this property one cannot prove that there exist functions not approximable by nomographable functions. This is explained by the fact that the indicated property is "unstable" in the sense that if in some braid there are several curves with an order of arrangement of intersection points which, in accordance with this property, is impossible for straight lines, then in an "arbitrarily close" braid of curves there may already be no such order of arrangement of intersection points. Below we shall give an example of a function $z = f(x, y)$ to which there corresponds a braid that is not only nonrectifiable, but also not approximable by rectifiable braids. It then follows from the lemma that all functions sufficiently close to $z = f(x, y)$ are not nomographable*. The construction of this example is based on a property of straight lines in the plane, established below, which is already "stable" in the sense indicated above.

Fig. 1

2. Let x_1, y_1, and z_1 be oriented straight lines in the affine plane (i.e. straight lines with a direction prescribed on each of them). If they determine a triangle or a half-strip**, then the choice of a definite order of the lines x_1, y_1, z_1 determines an orientation of the contour of this triangle or half-

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Fig. 2

Figure 2: Fig. 2

strip, and consequently also a new orientation of each of the lines x_1, y_1, z_1, which may or may not coincide with the prescribed orientation of these lines.

By the **index** of a triple of oriented straight lines x_1, y_1, z_1, taken in the indicated order, we shall mean the number equal to:

0, if the lines x_1, y_1, and z_1 pass through one point or are parallel;

1, if they determine a triangle or half-strips and the number of disagreements of orientation is even;

-1, if they determine a triangle or half-strips and the number of disagreements is odd.

Consider in the plane three straight lines x_1, y_1, and z_1 passing through one point O. Denote the six rays then obtained by the symbols $x_0^1, y_0^1, z_0^1, x_0^2, y_0^2, z_0^2$. We shall say that the six lines $x_-, y_-, z_-, x_+, y_+, z_+$ form a **hexagon** E with center O, diagonals x_0, y_0, and z_0, and initial ray z_0^2, if the points of intersection

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* We emphasize: all continuous functions sufficiently close to the function $z = f(x, y)$ and monotone in each of the variables are nonnomographable.

** In the latter case the lines x_1, y_1, z_1 determine two half-strips, but it is easy to see that the definition of the index of a triple of lines given below does not depend on which of the two half-strips we take.

of the lines y_+ and z_-, z_- and x_+, x_+ and y_-, y_- and z_+, and, finally, z_+ and x_- lie on those rays on which they are indicated in Fig. 1, and if the line y_+ does not intersect the line y_0 on the segment NO and the line x_- does not intersect the line x_0 on the segment MO. The lines x_-, y_-, z_-, x_+, y_+, z_+ will be called the **sides** of the hexagon E.

To each hexagon E we assign a number $\operatorname{ind} E$—the index of this hexagon, equal to $0, 1$, or -1. Orient the lines x_- and y_+ so that, if we look along any of these lines in the chosen direction, the point O remains on the left, and orient the line z_0 so that, if we look along this line in the chosen direction, the rays x_0^2 and y_0^2 remain on the left. By definition put

$$\operatorname{ind} E = \operatorname{ind}(x_-, y_+, z_0).$$

Fig. 2

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Now consider three hexagons E_1, E_2, and E_3, having common center O, common diagonals, and a common initial ray. The sides of the hexagon E_i $(i = 1, 2, 3)$ will be denoted by the symbols $x_{-i}, y_{-i}, \dots, z_{+i}$. We define the **linking coefficient** $k(E_1, E_2, E_3)$ of the hexagons E_1, E_2, E_3 (the order is essential) as follows. Orient all sides of these three hexagons in the same way as the lines x_- and y_+ were oriented above.

Put: $k(E_1, E_2, E_3) = 1$, if the indices of all triples of lines entering into the symbolic determinants

$$\begin{vmatrix} x_{-1} & y_{-1} & z_{-1} \\ x_{-2} & y_{-2} & z_{-2} \\ x_{+3} & y_{+3} & z_{+3} \end{vmatrix}, \qquad \begin{vmatrix} x_{-1} & y_{-1} & z_{-1} \\ x_{-1} & y_{-1} & z_{-1} \\ x_{+3} & y_{+3} & z_{+3} \end{vmatrix}, \qquad \begin{vmatrix} x_{-2} & y_{-2} & z_{-2} \\ x_{-2} & y_{-2} & z_{-2} \\ x_{+3} & y_{+3} & z_{+3} \end{vmatrix}$$

with plus sign are nonnegative, and the indices of all triples of lines entering into these determinants with minus sign are nonpositive; if, furthermore, the indices of all triples of lines entering into the determinants

$$\begin{vmatrix} x_{+1} & y_{+1} & z_{+1} \\ x_{+2} & y_{+2} & z_{+2} \\ x_{-3} & y_{-3} & z_{-3} \end{vmatrix}, \qquad \begin{vmatrix} x_{+1} & y_{+1} & z_{+1} \\ x_{+1} & y_{+1} & z_{+1} \\ x_{-3} & y_{-3} & z_{-3} \end{vmatrix}, \qquad \begin{vmatrix} x_{+2} & y_{+2} & z_{+2} \\ x_{+2} & y_{+2} & z_{+2} \\ x_{-3} & y_{-3} & z_{-3} \end{vmatrix}$$

with plus sign are nonpositive, and the indices of triples of lines entering into these determinants with minus sign are nonnegative, and if, finally, at least one of these 24 indices is different from zero. (The expansions for the determinants are written according to the usual rules for computing determinants, with in each triple the first line taken from the first row, the second from the second row, and the third from the third row.)

$k(E_1, E_2, E_3) = -1$, if in the preceding definition one everywhere replaces the word "nonnegative" by the word "nonpositive," and conversely.

$k(E_1, E_2, E_3) = 0$, if the indices of all 24 triples of lines entering into the preceding definition are equal to zero.

If $k(E_1, E_2, E_3)$ is defined, we shall say that the hexagons E_1, E_2, E_3 (in this order) are **interlaced**.

Theorem 1 (The theorem on three hexagons). *Let the hexagons E_1, E_2, and E_3 be interlaced. Then, if $\operatorname{ind} E_1 \cdot \operatorname{ind} E_2 = -1$, the equality holds*

$$k(E_1, E_2, E_3) = \operatorname{ind} E_3.$$

Corollary. *There does not exist a drawing consisting of three sevens of rectilinear segments $x_{-3}, x_{-2}, \dots, x_{+3}, \dots, z_{+3}$, whose points of intersection are arranged in the same order as the points of intersection of the corresponding lines in Fig. 2.*

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Indeed, the lines with index 1 would form a hexagon E_1 with center O and diagonals x_0, y_0, z_0. In exactly the same way the lines with index 2 would form a hexagon E_2, and the lines with index 3 a hexagon E_3 with the same center and the same diagonals. Using the definition of the coefficient of interlacing, we easily find $k(E_1, E_2, E_3) = -1$. At the same time, evidently, $\operatorname{ind} E_1 = -1$, $\operatorname{ind} E_2 = 1$, and $\operatorname{ind} E_3 = 1 \neq k(E_1, E_2, E_3)$, which contradicts Theorem 1.

3. Consider the function

$$p(t) = \begin{cases} -\frac{1}{12}(t-1)^7 + \frac{7}{12}(t-1) + \frac{1}{2}, & \text{for } 0 \le t \le 2, \\ 1, & \text{for } t > 2, \end{cases}$$

and $p(t) = p(-2t)$ for $t < 0$.

Theorem 2. *The interlacing corresponding to the function*

$$z = f(x, y) \equiv x + y - 1.1\, p(x)p(y)p(x+y) -$$

$$- 0.0001\, xy(x-2)(x-3)(y+1)\left(y - \frac{3}{2}\right),$$

in the square $R : |x| \le 3.5,\ \ |y| \le 3.5$*, cannot be approximated by rectifiable interlacings.*

Indeed, computing the values of the function $z = f(x, y)$ at the points of intersection of the lines $x = i$ and $y = j$ $(i, j = 0, \pm1, \pm2, \pm3)$, we verify that the level lines $z = 3,\ z = 2, \ldots,\ z = -3$ intersect these lines in the order indicated in Fig. 2. From the corollary to Theorem 1 it then follows that the interlacing corresponding to the function $z = f(x, y)$ is non-rectifiable. It can be shown that all interlacings sufficiently close to it will also be non-rectifiable.

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Note: Figure translations are in progress. See original paper for figures.

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