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Abstract

Full Text

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ASYMPTOTIC DISTRIBUTION AND ERGODIC PROPERTIES OF INTEGER POINTS ON A ONE-SHEETED HYPERBOLOID

(Presented by Academician I. M. Vinogradov on 28 VI 1960)

§ 1. We consider the surface of a one-sheeted hyperboloid H in Cartesian coordinates a, b, c ,

$$b^2 - ac = D, \quad D > 0. \quad (1)$$

The number D is assumed to be an integer, odd, and not a perfect square. On the surface H we single out the classical reduction domain Δ

$$b^2 - ac = D, \quad 0 < b < \sqrt{D}, \quad \sqrt{D} - b < |a| < \sqrt{D} + b. \quad (2)$$

By Δ_1 we shall denote a finitely connected figure with piecewise smooth boundary, contained in Δ . By the symbol $V(\Delta_1)$ we shall mean the volume of the cone with base Δ_1 and vertex at the origin. By the symbol $A(\Delta_1)$ we shall mean the hyperbolic solid angle, i.e. $A(\Delta_1) = V(\Delta_1)/D^{3/2}$. By $T(\Delta_1)$ we denote the number of integer points satisfying the condition $\gcd(a, 2b, c) = 1$ and lying on the surface Δ_1 .

Suppose that there exists a prime number p such that $(D/p) = +1$. Then the following holds.

Theorem 1.

$$T(\Delta_1) = \frac{A(\Delta_1)}{A(\Delta)} T(\Delta) (1 + \eta(p, D));$$

$$\eta(p, D) \rightarrow 0 \quad \text{for fixed } p, A(\Delta_1) \text{ and as } D \rightarrow \infty.$$

Theorem 1 is proved by a method which enabled Yu. V. Linnik, in particular, to prove the analogous theorem for the case of a two-sheeted hyperboloid.

§ 2. The essential difference between the reduction domains of the one-sheeted and two-sheeted hyperboloids does not permit a trivial transfer of the results of ⁽¹⁾ to the case of the one-sheeted hyperboloid. In particular, the theorem on cycles turned out to be essential in the proof of Theorem 1. We formulate it in terms of Gauss composition of binary quadratic forms.

Denote by A_1, A_2, B classes of forms of determinant $D > 0$; \sqrt{D} is irrational. Here $A_1 \cdot B = A_2$ (in the sense of composition). Denote by n_1, n_2 the number of representatives (in the reduction domain) of quadratic forms corresponding respectively to the classes A_1, A_2 , and let $0 < a = \min(B)$. Then there holds the

Theorem on cycles 1.

$$\frac{n_1}{n_2} < c \log(1 + a), \quad \text{where } c \text{ is an absolute constant.}$$

By making the reasoning more complicated and applying considerations of a geometric character, one can prove a stronger theorem.

Theorem on cycles 2.

$$\frac{n_1}{n_2} < \text{const} \cdot \log[1 + \log(1 + a)],$$

or, as a consequence, for $D > 3$

$$\frac{n_1}{n_2} < \text{const} \cdot \log \log D.$$

§ 3. Denote by $L = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$ a matrix with zero trace; a, b, c are integers; $\gcd(a, 2b, c) = 1$; $b^2 - ac = D > 0$; D is nonsquare, \sqrt{D} is irrational. Further, denote by R a matrix of the second order with integer components.

Let $\det R = p^k$, $(D/p) = +1$. Then there exists such a matrix R , with $\det R = p^k$, that $RLR^{-1} = L'$ will be integral and $\gcd(a', 2b', c') = 1$, and moreover $L' \in \Delta$. The n -fold rotation of the form RLR^{-1} forms an abstract flow. Note that R is chosen nonuniquely, and therefore it is essential to specify the direction of the flow.

It is very important that there exists a rule of selection for each $L \in \Delta$ of a single matrix R from the set of R_i for which $R_{iLR}i^{-1} = L'$ ($i = 0, 1, 2, \dots$), $L' \in \Delta$, L' integral. In this connection we shall denote an operation of the type RLR^{-1} by \mathcal{L} , and the image L' by $\mathcal{L}L$; the n -fold operation will be denoted by $\mathcal{L}^{(n)}$, and accordingly the image by $\mathcal{L}^{(n)}L$. Further, by $f_{\Delta_1}(L)$ we shall denote the characteristic function of the set of primitive integral points. Then the following holds:

Theorem 2 (ergodic).

$$\frac{1}{s} \sum_{n=1}^s f_{\Delta_1}(\mathcal{L}^{(n-1)}L) = \frac{A(\Delta_1)}{A(\Delta)} (1 + \eta(p^k, \Delta_1, D))$$

for all images of primitive points L , with the possible exception of $o(T(\Delta))$; $s \geq c_0 \log D$, and $\eta(p^k, \Delta_1, D) \rightarrow 0$ for fixed p^k, Δ_1 and $D \rightarrow \infty$.

Let \mathfrak{M} be any set of primitive integral points lying in the domain Δ . We shall denote by $\mathcal{L}^{(n)}\mathfrak{M}$ the set of primitive integral points of the domain Δ to which these points “flow” after an n -fold transformation \mathcal{L} ; $M(\mathfrak{M})$ is the number of points of \mathfrak{M} ; $M(\mathcal{L}^{(n)}\mathfrak{M} \cap \Delta_1)$ is the number of points of the set of images $\mathcal{L}^{(n)}\mathfrak{M}$ (counting also overlaps), whose images lie in the set Δ_1 . Then the following holds:

Theorem 3. Let $n = 0, 1, 2, \dots$, $s \geq c_1 \log D$, $M(\mathfrak{M}) > \varepsilon_0 T(\Delta)$ ($\varepsilon_0 > 0$ is any fixed number). Then for all indices n , with the possible exception of $s \cdot o(1)$ ($D \rightarrow \infty$) such indices, we have

$$M(\mathcal{L}^{(n)}\mathfrak{M} \cap \Delta_1) = \frac{A(\Delta_1)}{A(\Delta)} M(\mathfrak{M})(1 + \eta(p^k, \Delta_1, \varepsilon_0, D)),$$

where $\eta(p^k, \Delta_1, \varepsilon_0, D) \rightarrow 0$ for fixed $p^k, \Delta_1, \varepsilon_0$ and $D \rightarrow \infty$.

For all the theorems we take

$$A(\Delta) = \frac{4}{3}(2 \log 2 - 1).$$

It should be noted that, in contrast to the case of the sphere ⁽²⁾ and the two-sheeted hyperboloid, Theorem 1 (on the asymptotic distribution) does not follow from Theorem 3.

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CITED LITERATURE

1. Yu. V. **Linnik**, Vestn. LGU, No. 2, 3 (1955); No. 5, 3 (1955); No. 8, 15 (1955).
2. Yu. V. **Linnik**, Matem. sborn., 43 (85), 2, 257 (1957).

Note: Figure translations are in progress. See original paper for figures.

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