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Abstract

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I. S. Ponizovskii

ON HOMOMORPHISMS OF COMMUTATIVE SEMIGROUPS

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The problem of determining all homomorphisms of an arbitrary semigroup is still far from being solved. At present the solution of this problem is known only for certain, though interesting, classes of semigroups (L. M. Gluskin ([1]) described the homomorphisms of completely simple semigroups, Preston ([2]) the homomorphisms of inverse semigroups, and A. I. Mal' cev ([3,4]) the homomorphisms of symmetric and matrix semigroups). In the present note it is shown how the description of all homomorphisms of a commutative semigroup of fairly general type can be reduced to the description of homomorphisms of commutative semigroups which are called quasinilpotent in this note. Together with Tamura' s results ([5]), this leads, for example, to a description of all homomorphisms of an arbitrary finite commutative semigroup.

1. Let $\mathfrak{A}$ be a commutative semigroup with zero 0.

We shall call a system $A = [\mathfrak{A}_\alpha]$ of subsemigroups, totally ordered by decrease (where α ranges over the set of ordinal numbers from 1 to μ), a **decreasing ideal series** of $\mathfrak{A}$ if the following conditions are satisfied: a) the intersection of any collection of semigroups from A belongs to A; b) $\mathfrak{A}_1 = \mathfrak{A}$, $\mathfrak{A}_\mu = 0$; c) $\mathfrak{A}_{\alpha+1}$ is an ideal of $\mathfrak{A}_\alpha$ $(1 \leq \alpha < \mu)$.

The factor semigroups $\overline{\mathfrak{A}}_\alpha = \mathfrak{A}_\alpha / \mathfrak{A}_{\alpha+1}$ will be called the **factors** of A. The number μ will be called the **length** of the given series. A **principal decreasing ideal series** of $\mathfrak{A}$ will mean a decreasing ideal series $A = [\mathfrak{A}_\alpha]$ $(1 \leq \alpha \leq \mu)$ of the semigroup $\mathfrak{A}$, in which each member $\mathfrak{A}_{\alpha+1}$ is a maximal ideal of $\mathfrak{A}$ contained in $\mathfrak{A}_\alpha$ and distinct from $\mathfrak{A}_\alpha$ (i.e., if I is an ideal of $\mathfrak{A}$ and $\mathfrak{A}_\alpha \supset I \supset \mathfrak{A}_{\alpha+1}$, then either $I = \mathfrak{A}_\alpha$, or $I = \mathfrak{A}_{\alpha+1}$).

Let ν be an ordinal number. Define $\mathfrak{A}^\nu$ as follows. Put $\mathfrak{A}^1 = \mathfrak{A}$. Suppose that $\mathfrak{A}^\lambda$ have already been defined for all $\lambda < \nu$. Then put

$$\mathfrak{A}^\nu = \bigcap_{\lambda<\nu} \mathfrak{A}^\lambda, \quad \text{if } \nu \text{ is a limit ordinal,}$$

$$\mathfrak{A}^\nu = \mathfrak{A}^{\nu-1} \cdot \mathfrak{A}, \quad \text{if } \nu \text{ is not a limit ordinal.}$$

We shall call a semigroup $\mathfrak{A}$ **quasinilpotent** if, for some ordinal number τ, $\mathfrak{A}^\tau = 0$. If $\mathfrak{A}^n = 0$, where n is finite, then we shall call $\mathfrak{A}$ **nilpotent**. The semigroup $\mathfrak{A}$ is quasinilpotent if and only if $\mathfrak{A}$ has a decreasing ideal series all of whose factors are nilpotent.

2. The following proposition is the basis of the proposed description of homomorphisms of commutative semigroups.

Theorem 1. *Let $\mathfrak{A}$ be a commutative semigroup, $\mathfrak{A}^2 = \mathfrak{A}$, and suppose that $\mathfrak{A}$ has a principal decreasing ideal series of length $\mu \le \omega^*$. Then $\mathfrak{A}$ has a decreasing ideal series $A = [\mathfrak{A}_\alpha]$ such that:*

a) all members of A are ideals of $\mathfrak{A}$; b) each factor $\overline{\mathfrak{A}}_\alpha$ of the series A contains

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* By ω is denoted the first infinite ordinal number.

ideal $\overline{\mathfrak{B}}_\alpha$, which is a quasi-nilpotent semigroup, and moreover $\overline{\mathfrak{G}}_\alpha = \overline{\mathfrak{A}}_\alpha \setminus \overline{\mathfrak{B}}_\alpha$ is a group; c) the identity of $\overline{\mathfrak{G}}_\alpha$ is the identity of $\overline{\mathfrak{A}}_\alpha$.

3. Let the semigroup $\mathfrak{A}$ satisfy the conditions of Theorem 1. By Theorem 1, $\mathfrak{A}$ has a series

$$\mathfrak{A} = \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_\sigma = 0 \qquad (1)$$

such that assertions a), b), c) of Theorem 1 hold for it. In particular, each factor $\overline{\mathfrak{A}}_\alpha = \mathfrak{A}_\alpha / \mathfrak{A}_{\alpha+1}$ $(1 \le \alpha < \sigma)$ of the series (1) has an identity. This means that in $\mathfrak{A}_\alpha \setminus \mathfrak{A}_{\alpha+1}$ there exists an element e_α that is an identity for all elements of $\mathfrak{A}_\alpha \setminus \mathfrak{A}_{\alpha+1}$. Below, ω_α always denotes the ideal homomorphism of $\mathfrak{A}$ generated by the $\mathfrak{A}_\alpha$-term of the series (1).

Let $\overline{\varphi}_\alpha$ be a homomorphism of the factor $\overline{\mathfrak{A}}_\alpha = \mathfrak{A}_\alpha / \mathfrak{A}_{\alpha+1}$ of the series (1). If $x \in \mathfrak{A}$, then $xe_\alpha \in \mathfrak{A}_\alpha$ (e_α is the same as above) and $\omega_{\alpha+1}(xe_\alpha) \in \overline{\mathfrak{A}}_\alpha$. Therefore $\overline{\varphi}_\alpha \omega_{\alpha+1}(xe_\alpha)$ is meaningful. Put

$$\varphi_\alpha x = \overline{\varphi}_\alpha \omega_{\alpha+1}(xe_\alpha) \qquad (2)$$

for any $x \in \mathfrak{A}$. It is easy to verify that φ_α is a homomorphism of $\mathfrak{A}$.

The following proposition holds (the main one in the note).

Theorem 2. *Every homomorphism φ of a semigroup $\mathfrak{A}$ satisfying the conditions of Theorem 1 has the form* $*$

$$\varphi = \text{g.c.d. } \varphi_\alpha \qquad (1 \le \alpha < \nu),$$

where each homomorphism φ_α is obtained from some homomorphism $\overline{\varphi}_\alpha$ of the factor $\overline{\mathfrak{A}}_\alpha$ of the series (1) by formula (2).

Theorem 2 reduces the question of describing the homomorphisms of the semigroup $\mathfrak{A}$ to the question of describing the homomorphisms of the factors of the series (1). The structure of these factors is described by Theorem 1.

Theorem 3 shows how, knowing the homomorphisms of quasi-nilpotent semigroups, one can describe the homomorphisms of the factors of the series (1), and hence completely describe the homomorphisms of $\mathfrak{A}$.

Theorem 3. *Let $\mathfrak{C}$ be a commutative semigroup, $\mathfrak{B}$ an ideal of $\mathfrak{C}$ that is a quasi-nilpotent semigroup, and $\mathfrak{G} = \mathfrak{C} \setminus \mathfrak{B}$ a group whose identity is an identity of $\mathfrak{C}$.*

Every homomorphism φ of the semigroup $\mathfrak{C}$ induces homomorphisms π, ρ, respectively, of the ideal $\mathfrak{B}$ and the group $\mathfrak{G}$:

$$\begin{aligned} \varphi x &= \pi x \qquad \text{for } x \in \mathfrak{B}, \\ \varphi x &= \rho x \qquad \text{for } x \in \mathfrak{G}. \end{aligned} \tag{3}$$

Here only two cases are possible:

I. $\varphi\mathfrak{B} \cap \varphi\mathfrak{G}$ is empty; π, ρ are connected by the conditions: a) from $\pi x = \pi y$ $(x, y \in \mathfrak{B})$ it follows that $\pi(xz) = \pi(yz)$ for any $z \in \mathfrak{G}$; b) from $\rho z = \rho u$ $(z, u \in \mathfrak{G})$ it follows that $\pi(xz) = \pi(xu)$ for any $x \in \mathfrak{B}$.

II. In $\mathfrak{B}$ there is a normal subsemigroup $\mathfrak{N}$ such that $\pi\mathfrak{N} = \rho\mathfrak{G}$, and from $\rho z = \pi N$ $(z \in \mathfrak{G},\ N \in \mathfrak{N})$ it follows that $\pi(xz) = \pi(xN)$ for any $x \in \mathfrak{B}$; in this case $\varphi\mathfrak{C} = \varphi\mathfrak{B}$.

Conversely, if π, ρ are respectively homomorphisms of $\mathfrak{B}, \mathfrak{G}$ and either I a), b), or II is satisfied, then formulas (3) determine a homomorphism φ of the semigroup $\mathfrak{C}$, if one sets

$$\varphi x \cdot \varphi z = \varphi(xz) \qquad (x \in \mathfrak{B},\ z \in \mathfrak{G}).$$

4. If $\mathfrak{A}^2 \neq \mathfrak{A}$, then, adjoining to $\mathfrak{A}$ an identity externally, we obtain a semigroup $\mathfrak{A}'$, and ${\mathfrak{A}'}^2 \neq \mathfrak{A}'$. The description of the homomorphisms of $\mathfrak{A}$ is equivalent

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∗ By g.c.d. φ_α $(1 \leq \alpha < \nu)$ is denoted the common greatest divisor of the homomorphisms φ_α.

in this case, to the description of the homomorphisms of $\mathfrak{A}'$. Therefore it is enough to be able to describe the homomorphisms of all such semigroups $\mathfrak{A}$ for which $\mathfrak{A}^2 = \mathfrak{A}$; this is done in Theorem 2.

5. A finite quasi-nilpotent semigroup is nilpotent. The structure of finite nilpotent (not necessarily commutative) semigroups is described in (5);

from this description one very easily obtains a description of all homomorphisms of finite nilpotent semigroups. Together with Theorems 2 and 3 this gives a description of all homomorphisms of an arbitrary finite commutative semigroup.

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Note: Figure translations are in progress. See original paper for figures.