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Abstract

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On Certain Properties of Solutions of Mixed Problems for a Parabolic Equation with Discontinuous Coefficients

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Recently, numerous works have been devoted to the consideration of boundary-value problems for a second-order equation with discontinuous coefficients (see, for example, ([1–3])). In the present note a priori estimates are derived for solutions of various mixed problems for a parabolic equation with discontinuous coefficients, on the basis of which uniqueness and well-posedness theorems are proved. We note that we use methods developed by O. A. Oleinik in ([4]).

In the cylindrical domain $Q = \Omega \times (0, T)$ (Ω is a domain of the space $x = (x_1, \dots, x_n)$, bounded by the closed surface S) we consider the parabolic equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = 0, \qquad (1)$$

where, for arbitrary real λ_i, in $\overline{Q}$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\lambda_i\lambda_j \geq m \sum_{i=1}^{n} \lambda_i^2, \qquad m = \text{const} > 0, \qquad c(x,t) \leq 0.$$

The coefficients $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$ are assumed to be sufficiently smooth functions everywhere in the domain Q, except, perhaps, at points of a finite number of n-dimensional cylindrical manifolds $\Gamma_k = S_k \times (0, T)$ with generators parallel to the t-axis, where a_{ij}, b_i, c may have discontinuities of the first kind.

The manifolds Γ_k divide Q (respectively, S_k divide Ω) into a finite number of domains $Q_k = \Omega_k \times (0, T)$. The manifolds serving as the boundary of Q_k and Q_l (Ω_k and Ω_l) and not coinciding with $\Gamma = S \times (0, T)$ will be denoted by Γ_{kl} (respectively, S_{kl}). In what follows we shall assume that the Γ_{kl} have no common points with Γ and do not intersect one another.

Let a_{ij}, b_i, c have limiting values on both sides of Γ_{kl}, which we shall denote by $a_{ij}^{(k)}$, $a_{ij}^{(l)}$, etc. Let the boundary surfaces Γ and Γ_{kl} belong to the Lyapunov class. We investigate the properties of a solution $u(x,t)$, continuous in $\overline{Q}$, of the equation

$$Lu = f(x,t), \qquad (x,t) \in Q_k, \qquad (2)$$

with the boundary condition on Γ

$$l(u) \equiv a(x,t)\,du/dN + b(x,t)u\big|_{\Gamma} = \varphi(x,t); \tag{3}$$

with the initial condition

$$u(x,0) = F(x), \qquad x \in \overline{\Omega}; \tag{4}$$

with boundary conditions on Γ_{kl}

$$\alpha_k(x,t)\partial u/\partial N_k + \alpha_l(x,t)\partial u/\partial N_l\big|_{\Gamma_{kl}} = h_{kl}(x,t), \tag{5}$$

$$u(x,t)\big|_{\Gamma_{kl}-0} = u(x,t)\big|_{\Gamma_{kl}+0}. \tag{6}$$

where $F(x)$ satisfies the compatibility condition

$$a(x,0)\partial F(x)/\partial N + b(x,0)F(x) = \varphi(x,0). \tag{7}$$

$$\frac{\partial N}{\partial N_k} = \sum_{i,j=1}^{n} a_{ij}^{(k)} \cos(n_k, x_i)\frac{\partial}{\partial x_j}$$

is the derivative with respect to the conormal N_k, where n_k is the normal to Γ_{kl}, interior with respect to the domain Q_k; $\partial/\partial N$ is the derivative with respect to the conormal to Γ;

$$\alpha_k(x,t) \geq \alpha > 0 \qquad \text{for } (x,t) \in \Gamma_k; \tag{8}$$

$$a(x,t) \geq 0, \qquad b(x,t) \leq 0, \qquad a^2(x,t) + b^2(x,t) > 0 \qquad \text{for } (x,t) \in \Gamma. \tag{9}$$

For auxiliary purposes we consider cylindrical domains $Q^{(l)} = \Omega^{(l)} \times (0,T)$ $(l = 1,2)$ of two kinds: 1) the domain $Q^{(1)}$ has two lateral boundary surfaces $\Gamma = S \times (0,T)$ (external) and $\gamma = s \times (0,T)$ (internal); 2) the domain $Q^{(2)}$ has one lateral boundary surface $\Gamma = S \times (0,T)$. Consider functions $R_\delta^{(l)}(x)$ $(l = 1,2)$, continuous together with their derivatives of sufficiently high order in the closed domain $\overline{\Omega}^{(l)}$ $(l = 1,2)$, possessing the properties: $0 \leq R_\delta^{(l)}(x) < A$, $x \in \Omega^{(l)}$ $(l = 1,2)$; $\frac{\partial}{\partial n}R_\delta^{(l)}(x)\big|_s < 0$ $(l = 1,2)$; $\frac{\partial}{\partial n}R_\delta^{(1)}(x)\big|_s < 0$, where n is the interior normal to S (s); $R_\delta^{(l)}(x) \equiv 0$ for $\rho(x,S) > \delta$ $(l = 1,2)$ and for $\rho(x,s) > \delta$ $(l = 1)$;

$R_\delta^{(l)}(x)\big|_S = R_\delta^{(1)}(x)\big|_s = A$ $(l = 1, 2)$, where $\rho(x, S)$ is the distance from $x \in \Omega^{(l)}$ to S; $\delta > 0$ and $A > 0$ are constants, with $\rho(S, s) > \delta$.

Condition A. Let the coefficients of the operator Lu (1) in the domain $Q^{(l)}$ $(l = 1, 2)$, where

$$c(x, t) \leq 0, \tag{10}$$

and the boundary surfaces Γ and γ be so smooth that there exists, continuous together with its first derivatives with respect to x_k $(k = 1, 2, \dots, n)$ in the closed domain $\overline{Q}^{(l)}$ $(l = 1, 2)$, a solution $V_l(x, t)$ $(l = 1, 2)$ of the mixed problem

$$LV_l = 0, \qquad (x, t) \in Q^{(l)} \quad (l = 1, 2); \tag{11}$$

$$V_l\big|_\Gamma = V_1\big|_\gamma = A > 0 \quad (l = 1, 2); \tag{12}$$

$$V_l(x, 0) = R_\delta^{(l)}(x), \qquad x \in \overline{\Omega}^{(l)} \quad (l = 1, 2), \tag{13}$$

where $R_\delta^{(l)}(x)$ $(l = 1, 2)$ are the smooth functions constructed above.

Remark 1. Condition A holds if, for a_{ij}, b_i, c, Γ, and γ, the conditions of the existence theorems of T. D. Venttsel' (5) and A. Friedman (7) are satisfied.

Lemma 1. If condition A is fulfilled, then for the solution $V_l(x, t)$ of problem (11)–(13) the inequalities

$$0 \leq V_l(x, t) < A, \qquad (x, t) \in Q^{(l)}; \tag{14}$$

$$0 \leq V_l(x, t) \leq A, \qquad (x, t) \in \overline{Q}^{(l)}; \tag{15}$$

$$-\frac{\partial V_l}{\partial n}\bigg|_\Gamma < -r, \qquad \frac{\partial V_1}{\partial n}\bigg|_\gamma < -r, \qquad l(V_1)\big|_\Gamma < 0 \quad (l = 1, 2), \tag{16}$$

hold, where $r > 0$ is a constant.

Lemma 1 is proved with the aid of theorem 1 of R. Vyborny (6).

Theorem 1. If $u(x, t)$ is a solution of equation (1), continuous in $\overline{Q}$, satisfying conditions (5), (6), and

$$l(u)\big|_\Gamma = 0, \tag{17}$$

$$u(x,0) = 0, \tag{18}$$

having derivatives $\partial u/\partial N$ on Γ and $\partial u/\partial N_k$, $\partial u/\partial N_l$ on Γ_{kl}, where the surfaces Γ and Γ_{kl} are so smooth that Lemma 1 applies, then everywhere in $\overline{Q}$ the estimate holds

$$|u(x,t)| \leq \frac{A}{r^{\alpha}} \max_{k,l} \max_{(x,t)\in\Gamma_{kl}} |h_{kl}(x,t)|, \tag{19}$$

where A, r, α are the constants from (8), (15), and (16).

The proof is based on Lemma 1, on the validity of the maximum principle for the parabolic equation (1) in each of the closed subdomains $\overline{Q}_k$, and on Theorem 1 of R. Viborny (^{6}).

Remark 2. For the solution of the homogeneous first boundary-value problem in the case of an elliptic equation, an a priori estimate analogous to (19) was obtained by I. A. Shishmarev (^{3}).

From Theorem 1 one immediately obtains the uniqueness theorem for the solution of problem (2)–(6).

Theorem 2. If the coefficients of equation (2) and the boundary surfaces Γ and Γ_k satisfy the conditions under which Lemma 1 is valid, and if the functions $f(x,t)$, $F(x)$, $\partial F(x)/\partial x_i$, $\varphi(x,t)$, and $h_{kl}(x,t)$ in (2)–(6) are continuous in their domains of definition Q, Ω, Γ, and Γ_{kl}, respectively, and the conditions (7)–(9) are satisfied, then the mixed problem (2)–(6) has at most one solution $u(x,t)$, continuous in the closed domain $\overline{Q}$, twice continuously differentiable in the open domains Q_k, and having derivatives along the inner conormals to the boundary surfaces Γ and Γ_k.

Lemma 2. If (10) is satisfied, $u(x,t)$ is a solution of problem (2)–(6) continuous in $\overline{Q}$, where $f(x,t) \geq 0$ ($f(x,t) \leq 0$), $\varphi(x,t) \geq 0$ ($\varphi(x,t) \leq 0$), and $h_{kl}(x,t) \geq 0$ ($h_{kl}(x,t) \leq 0$), then $u(x,t)$ cannot attain a positive maximum (negative minimum) on any of the boundary surfaces Γ, Γ_{kl}.

Using Theorem 1 and Lemma 2 makes it possible to prove the following theorems.

Theorem 3. Let $u(x,t)$ be a solution of problem (2), (17), (18), (5), and (6), continuous in $\overline{Q}$. If

$$c(x,t) < 0, \tag{20}$$

then everywhere in $\overline{Q}$ the estimate holds

$$|u(x,t)| \le \frac{\max\limits_{(x,t)\in\overline{Q}} |f(x,t)|}{\min\limits_{(x,t)\in\overline{Q}} |c(x,t)|} + \frac{A}{r^\alpha} \max_k \max_{(x,t)\in\Gamma_k} |h_k(x,t)| \equiv B(f,h),$$

where A, r, α are the constants introduced above.

Theorem 4. Suppose that for arbitrary functions $\Psi_1(x,t)$, $\Psi_2(x,t)$ of the class $C^k(\overline{Q})$ $(k \ge 0)$ there exists a solution of equation (1), continuous in $\overline{Q}$ ($c(x,t)$ satisfies (20)), satisfying the conditions (5) with $h_k(x,t) \equiv 0$ and $l(u)|_\Gamma = \Psi_1(x,t)$, $u(x,0) = \Psi_2(x)$.

If $a(x,t)$ and $b(x,t)$ satisfy (9) and belong to the same class $C^{(k)}\Gamma$, then the solution $u(x,t)$, continuous in $\overline{Q}$, of problem (2), (3), (5), (6), (18) satisfies the inequality

$$|u(x,t)| \le B(f,h) + K_1 \frac{\max\limits_{(x,t)\in\Gamma} |\varphi(x,t)|}{\min\limits_{(x,t)\in\Gamma} (|a(x,t)| + |b(x,t)|)} \times$$

$$\times \left\{ 1 + K_2 \left[\frac{2A}{r^\alpha} \sum_{j,i=1}^{n} \max_{(x,t)\in\overline{Q}} |a_{ij}(x,t)| \cdot \max_{(x,t)\in\Gamma_k} |\alpha_k(x,t)| + \right.\right.$$

$$\left.\left. + \frac{\max\limits_{(x,t)\in\overline{Q}} \left(\sum\limits_{i,j=1}^{n} |a_{ij}| + \sum\limits_{i=1}^{n} |b_i| + 1 \right)}{\min\limits_{(x,t)\in\overline{Q}} |c(x,t)|} \right] \right\} \equiv M(f,\varphi,h), \tag{21}$$

where K_1, K_2 do not depend on the coefficients of the equation or on the functions f, φ, h_k.

Theorem 5. If $u(x,t)$ is a solution of equation (1), continuous in $\overline{Q}$, with homogeneous boundary conditions (17) and (5) ($h_k \equiv 0$) and with initial condition (4), where $F(x)$ is continuous in $\overline{\Omega}$, and the compatibility conditions (7) ($\varphi \equiv 0$) are satisfied, then everywhere in $\overline{Q}$ the inequality

$$|u(x,t)| \le \max_{x\in\overline{\Omega}} |F(x)|$$

holds.

With the aid of Theorems 4 and 5 one proves

Theorem 6. If the conditions of Theorem 4 are satisfied and if, for the function $F(x)$, the mixed problem (1) with homogeneous boundary conditions (17) and (5) ($h_k(x,t) \equiv 0$) and with initial conditions (4) is solvable, then the solution $u(x,t)$ of problem (2)–(7), continuous in $\overline{Q}$, satisfies everywhere in $\overline{Q}$ the inequality

$$|u(x,t)| \leq \max_{x \in \Omega} |F(x)| + M(f, \varphi, h), \tag{22}$$

where $M(f, \varphi, h)$ is taken from inequality (21) of Theorem 4.

The estimates (21), (22) obtained make it possible to prove the continuous dependence of the solution of the mixed problem (2)–(7) on the coefficients (2), the boundary and initial conditions.

Theorem 7. Let $u(x,t)$, $\overline{u}(x,t)$ be two solutions, continuous in $\overline{Q}$, of problem (2)–(6) with the corresponding right-hand sides $\varphi, \overline{\varphi}, h_k, \overline{h}_k, F, \overline{F}$. Then the difference $|u(x,t) - \overline{u}(x,t)|$ in $\overline{Q}$ will be arbitrarily small if the moduli of the differences $|f - \overline{f}|$, $|\varphi - \overline{\varphi}|$, $|h_k - \overline{h}_k|$, $|F - \overline{F}|$ are sufficiently small.

Theorem 8. Let, for the solution $u(x,t)$, continuous in $\overline{Q}$, ($c < 0$), of the mixed problem (2)–(6), all the conditions of Theorem 6 be satisfied. Let $\overline{u}(x,t)$ be a solution, continuous in $\overline{Q}$, of the mixed problem (3)–(6) for the equation $\overline{Lu} = \overline{f}$ ($\overline{c} < 0$), where all corresponding coefficients and right-hand sides have been replaced by $\overline{a}_{ij}, \overline{b}_i, \overline{c}, \overline{a}, \overline{b}, \overline{\alpha}_k, \overline{\varphi}, \overline{f}$. Suppose that $\partial^2 u/\partial x_i \partial x_j$, $\partial u/\partial x^i$ are bounded in each of the domains Q_k; $\partial u/\partial N$, $\partial u/\partial N_k$ are bounded on Γ and Γ_k. Then the difference $|u(x,t) - \overline{u}(x,t)|$ everywhere in the domain $\overline{Q}$ will be arbitrarily small if the moduli of the differences of the corresponding coefficients of the equations, boundary conditions, right-hand sides, and initial data are sufficiently small.

Remark 3. All theorems and estimates will remain valid when (3) is replaced by the condition

$$a(x,t)\partial u/\partial \nu + b(x,t)u|_\Gamma = \varphi(x,t),$$

and (5) by the boundary conditions

$$\alpha_k(x,t)\partial u/\partial \nu_k + \alpha_l(x,t)\partial u/\partial \nu_l|_{\Gamma_{kl}} = h_{kl}(x,t),$$

where ν and ν_k are directions making acute angles with the normals $\mathbf{n}$ and $\mathbf{n}_k$ to the boundary surfaces Γ and Γ_k (interior with respect to Q and Q_k, respectively).

Remark 4. Analogous theorems can also be proved for moving lateral boundaries Γ and Γ_k, provided only that they do not intersect one another and that the normals to each of them are nowhere parallel to the t-axis.

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