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Abstract

Full Text

Mechanics

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ON NONLINEAR OSCILLATIONS IN A WEAK EXTERNAL FIELD DESCRIBED BY LAGRANGE EQUATIONS

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§ 1. Transformation of the Lagrange equations

1. Consider a system with n degrees of freedom:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k \qquad (k = 1, \dots, n). \tag{1}$$

We shall assume that the motion of the system without the forces Q_k

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{p}_k} - \frac{\partial L}{\partial q_k} = 0 \tag{2}$$

is known. We shall seek the solution of system (1) in the same form as the solution of system (2), regarding the arbitrary constants C in the solution of system (2) as functions of time and, in doing so, not leaving the region for which the indicated general solution of system (2) is defined. To determine these $2n$ functions $C(t)$ (or quantities equivalent to them), we apply a canonical transformation of the generalized coordinates q and momenta p ($p_k = \partial L(q, \dot{q}, t)/\partial \dot{q}_k$):

$$q_k = q_k(\tilde{q}, \tilde{p}, t), \qquad p_k = p_k(\tilde{q}, \tilde{p}, t), \tag{3}$$

under which the Hamilton equations equivalent to (2) retain their form. Then for the new canonical coordinates $\tilde{q}$ and momenta $\tilde{p}$ we obtain the system of equations

$$\frac{d\tilde{q}_k}{dt} = -\sum_i \left[Q_i \frac{\partial q_i}{\partial \tilde{p}_k} + \left(\frac{\partial L}{\partial q_i} - \frac{\partial p_i}{\partial t} \right) \frac{\partial q_i}{\partial \tilde{p}_k} - \left(\dot{q}_i - \frac{\partial q_i}{\partial t} \right) \frac{\partial p_i}{\partial \tilde{p}_k} \right], \tag{4}$$

$$(i, k = 1, \dots, n)$$

$$\frac{d\tilde{p}_k}{dt} = \sum_i \left[Q_i \frac{\partial q_i}{\partial \tilde{q}_k} + \left(\frac{\partial L}{\partial q_i} - \frac{\partial p_i}{\partial t} \right) \frac{\partial q_i}{\partial \tilde{q}_k} - \left(\dot{q}_i - \frac{\partial q_i}{\partial t} \right) \frac{\partial p_i}{\partial \tilde{q}_k} \right]. \tag{5}$$

2. If $\tilde{p}_k$ and $\tilde{q}_k$ are the constants of (2) (the case considered in ([1])), $\tilde{p}_k = \alpha_k,\ \tilde{q}_k = \beta_k$ (for example, $\alpha_k = p_k(t_0),\ \beta_k = q_k(t_0)$, where t_0 is an arbitrary instant of time), then the brackets in (4) and (5) are equal to zero, and we shall have

$$\frac{d\alpha_k}{dt} = \sum_i Q_i \frac{\partial q_i(\beta, \alpha, t)}{\partial \beta_k}, \qquad \frac{d\beta_k}{dt} = -\sum_i Q_i \frac{\partial q_i(\beta, \alpha, t)}{\partial \alpha_k}. \tag{6}$$

When $\partial L / \partial t = 0$, one of the pairs of constants may be taken to be the total energy of (2),

$$E \equiv H = \sum_k p_k \dot{q}_k - L$$

(equal to the work of the forces Q_k in (1))

and an arbitrary constant τ, additive to the time t ($\alpha_1 = \tau,\ \beta_1 = -E$):

$$\frac{dE}{dt} = \sum_i Q_i \dot{q}_i, \qquad \frac{d\tau}{dt} = -\sum_i Q_i \frac{\partial q_i}{\partial E}. \tag{7}$$

3. If the time t does not enter explicitly into the transformation (3), then equations (4) and (5) reduce to the system

$$\frac{d\tilde{q}_k}{dt} = -\sum_i Q_i \frac{\partial q_i}{\partial \tilde{p}_k} + \frac{\partial H}{\partial \tilde{p}_k}, \qquad \frac{d\tilde{p}_k}{dt} = \sum_i Q_i \frac{\partial q_i}{\partial \tilde{q}_k} - \frac{\partial H}{\partial \tilde{q}_k}. \tag{8}$$

In this case, for $\tilde{p}_k$ and $\tilde{q}_k$ one may take, for example, the action–angle variables (see § 3).

§ 2. **The case of one-dimensional oscillatory (rotatory) motion.** Suppose (2) (for $n = 1$ we denote $Q_k = Q,\ q_k = q$) has the solution

$$q = q(a, \psi), \qquad \dot{q} = \dot{q}(a, \psi), \qquad \psi = \omega(a)t + \varphi = \psi(a, q), \tag{9}$$

where a and φ are arbitrary constants, of which only a is determined by the total energy E ($E = E(a)$); ψ and $\omega(a)$ are the phase and the angular frequency of the oscillation (libration) or rotation. From (7) and (1) we obtain exact equations for the "amplitude" a and phase ψ*

$$\frac{da}{dt} = \frac{1}{E'(a)} Q\dot{q}, \qquad \frac{d\Phi}{dt} = \omega_1 - \frac{l_1}{m_1}\omega(a)\left[1 - \frac{1}{E'(a)} Q \frac{\partial q(a,\psi)}{\partial a}\right]. \tag{10}$$

In (10) the coefficients of Q are periodic functions of ψ with period 2π. To simplify (10) we use the basic idea of the averaging method ([2]).

1. Let $Q = Q(\theta_1, q, \dot{q}) \equiv Q(\theta_1 + 2\pi, q, \dot{q})$, where

$$\frac{d\theta_1}{dt} = \omega_1 \simeq \frac{l_1}{m_1}\omega$$

(Q is a periodic function of θ_1 and ψ with period 2π; l_1 and m_1 are relatively prime, generally speaking small, numbers). Introducing the relative generalized phase

$$\Phi = \theta_1 - \frac{l_1}{m_1}\psi,$$

we bring (10) to the standard form

$$\frac{da}{dt} = \frac{1}{E'(a)} Q\dot{q}, \qquad \frac{d\Phi}{dt} = \omega_1 - \frac{l_1}{m_1}\omega(a)\left\{1 - \frac{1}{E'(a)} Q \frac{\partial q(a,\psi)}{\partial a}\right\}. \tag{11}$$

Since

$$Q(\theta_1, q, \dot{q}) \equiv Q\left[\Phi + \frac{l_1}{m_1}\psi,\ q(a,\psi),\ \dot{q}(a,\psi)\right],$$

the right-hand sides of (11) are periodic functions of ψ with period $2\pi m_1$, and, for small Q ($Q \sim \varepsilon$), are small on such an interval $t \in \Delta(\varepsilon)$, in which the ratio

$$\delta(\varepsilon) = \left|\omega_1 - \frac{l_1}{m_1}\omega[a(t)]\right| / \omega_1 \ll 1$$

is small. If this interval is such that the period of the oscillation

$$T(a) = \frac{2\pi}{\omega(a)} \ll \Delta(\varepsilon),$$

then the first-approximation equations for a and Φ, valid on the interval $\Delta(\varepsilon)$, will be

$$\frac{da}{dt} = \frac{1}{E'(a)} \overline{Q\dot{q}} = \frac{1}{m_1 T(a) E'(a)} \oint_{m_1} Q\, dq, \tag{12}$$

$$\frac{d\Phi}{dt} = \omega_1 - \frac{l_1}{m_1}\omega(a)\left[1 - \frac{1}{E'(a)}\overline{Q\frac{\partial q(a,\psi)}{\partial a}}\right], \tag{13}$$

where $\overline{}$ and $\oint_{m_1}$ denote, respectively, the average and the integral over m_1 cycles of (2), with a and Φ fixed**.

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* For the special case of writing (1) in the form $\ddot{x} + f(x) = \varepsilon F$, the equations (2.214) of Yu. A. Mitropolsky ([3]) reduce, naturally, to (10).

** If Q can be represented in the form $Q = Q'(\theta_1)Q''(q,\dot{q})$, then integration by parts reduces (13) to the form:

$$\frac{d\Phi}{dt} = \omega_1 - \frac{l_1}{m_1}\omega(a)\left[1 + \frac{1}{E'(a)m_1\omega_1 T(a)}\oint_{m_1} Q''\frac{\partial}{\partial q}\left(Q'''\frac{\partial q(a,\psi)}{\partial a}\right)dq\right]$$

$$\left(Q''' = \int Q'(\theta_1)\,d\theta_1\right).$$

In the equations obtained, integration over m_1 cycles can be carried out when the right-hand sides of the equations are written piecewise-analytically; this makes it possible to study strongly nonlinear oscillations in a weak perturbing field.

2. If Q is a multiperiodic function of $\theta_1, \dots, \theta_s$ with period 2π in each θ_i, and

$$\frac{d\theta_1}{dt} = \omega_1 \simeq \frac{l_1}{m_1}\omega, \dots, \quad \frac{d\theta_s}{dt} = \omega_s \simeq \frac{l_s}{m_s}\omega,$$

then the averaging must be carried out over $M(m_1, \dots, m_s)$ cycles of oscillation, where M is the least common multiple of the numbers $m_1, \dots, m_s$

$$\left(\theta_i = \Phi\frac{\omega_i}{\omega_1} + \frac{l_i}{m_i}\psi, \quad \Phi = \theta_1 - \frac{l_1}{m_1}\psi\right).$$

3. Let Q depend on t in an arbitrary way*, $Q = Q(t, q, \dot{q})$. By the substitution $\psi = \frac{1}{r}(t - \Phi)$, where $r = \text{const} \simeq \frac{1}{\omega(a)}$, we obtain the first-approximation equations

$$\frac{da}{dt} = \frac{1}{E'(a)}\overline{Q\dot{q}}, \quad \frac{d\Phi}{dt} = 1 - r\omega(a)\left[1 - \frac{1}{E'(a)}\overline{Q\frac{\partial q(a,\psi)}{\partial a}}\right], \tag{14}$$

where the averaging with respect to t should be carried out over such an interval Δ in which

$$|1 - r\omega[a(t)]| \ll 1.$$

§ 3. General case of motion close to periodic motion.

1. Let (2) describe multiperiodic motion

$$q_k = q_k(w_k, I), \quad \dot{q}_k = \dot{q}_k(w_k, I), \quad w_k = \nu_k(I)t + h_k \quad (k = 1, \dots, n), \tag{15}$$

where $\nu(I)$ are the frequencies of libration or rotation; h, I are $2n$ arbitrary constants. Choosing the actions as the constants I

$$\left(I_k = \oint p_k \, dq_k\right)$$

and putting in (8) $\tilde{q}_k = w_k$, $\tilde{p}_k = I_k$, we obtain the exact system**

$$\frac{dI_k}{dt} = Q_k \frac{\partial q_k(w_k, I)}{\partial w_k} = \frac{Q_k \dot{q}_k}{\nu_k(I)}, \quad \frac{dw_k}{dt} = \nu_k - \sum_i Q_i \frac{\partial q_i(w_i, I)}{\partial I_k}, \tag{16}$$

in which the coefficients at the forces Q_k are periodic in w_k with period 1

$$(\nu_k(I) = \partial H(I)/\partial I_k) \, .$$

2. If the system (2) is m-fold degenerate, i.e. there exist m relations

$$\sum_{i=1}^{n} j_{ki} \nu_i = 0 \quad (k = 1, \dots, m),$$

where j_{ki} are integers (generally speaking, small), then by the substitution

$$w'_k = \sum_{i=1}^{n} j_{ki} w_i \quad (k = 1, \dots, m); \qquad w'_k = w_k \quad (k = m+1, \dots, n)$$

for the new variables w' we obtain the equations***

$$\frac{dw'_k}{dt} = -\sum_{i,s=1}^{n} j_{ki} Q_s \frac{\partial q_s}{\partial I_i}, \quad (k = 1, \dots, m), \tag{17}$$

$$\frac{dw'_k}{dt} = \nu_k(I) - \sum_{i=1}^{n} Q_i \frac{\partial q_i}{\partial I_k}, \quad (k = m+1, \dots, n). \tag{17a}$$

In (17a) all ν_k are linearly independent, while in (17) the right-hand sides are small.

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* The case when Q does not explicitly depend on t, $Q = Q(q, \dot{q})$, is considered below.

** For canonical systems with one degree of freedom

$$(Q = -\partial H(q, p, t)/\partial q)$$

the variables I, w were used in (6).

*** If

$$\sum_{i=1}^{n} j_{ki}\nu_i = \varepsilon P_k(I),$$

then equations (17), as well as (18a) (see below), will contain in their right-hand sides the small term $\varepsilon P_k(I)$ $(k = 1, \dots, m)$.

3. Let the forces Q_k be small and not contain t explicitly*. Then the equations of the first approximation will be (similar to the equations obtained when using normal coordinates in (7)):

$$\frac{dI_k}{dt} = \frac{1}{\nu_k(I)(2\pi)^{n-m}} \int_0^{2\pi} \cdots \int_0^{2\pi} Q_k \dot{q}_k \, d\psi'_{m+1} \cdots d\psi'_n \quad (k = 1, \dots, n), \tag{18}$$

$$\frac{d\omega'_k}{dt} = -\frac{1}{(2\pi)^{n-m}} \sum_{i,s=1}^{n} \int_0^{2\pi} \cdots \int_0^{2\pi} j\, Q_s \frac{\partial q_s}{\partial I_i} \, d\psi'_{m+1} \cdots d\psi'_n \quad (k = 1, \dots, m), \tag{18a}$$

$$\frac{d\omega'_k}{dt} = \nu - \frac{1}{(2\pi)^{n-m}} \sum_i \int_0^{2\pi} \cdots \int_0^{2\pi} Q_i \frac{\partial q_i}{\partial I_k} \, d\psi'_{m+1} \cdots d\psi'_n \quad (k = m+1, \dots, n). \tag{19}$$

Here $\dot{\psi}'_k = 2\pi\omega'_k$. The $n + m$ equations (18), (18a) contain on their right-hand sides $n + m$ unknown functions $I_1, \dots, I_n$; $\omega'_1, \dots, \omega'_m$ and can be integrated independently of equations (19).

4. If t enters explicitly into the forces Q_k, then, by the substitution $W_k = t - r_k\omega_k$, where $r_k = \text{const} \simeq 1/\nu_k(I)$, we obtain, analogously to (14),

$$\frac{dI_k}{dt} = \frac{1}{\nu_k(I)} \overline{Q_k \dot{q}_k}, \qquad \frac{dW_k}{dt} = 1 - r_k\nu_k(I) + \sum_i \left[\overline{Q_i \frac{\partial q_i}{\partial I_k}} \right]. \tag{20}$$

In (20), averaging with respect to t should be carried out over such an interval $t \in \Delta$ in which $|1 - r_k\nu_k[I(t)]| \ll 1$ for all k. The question of the accuracy of the equations of the first approximation requires a special investigation and is not considered here.

Remark. If instead of (15) the complete system of independent integrals (2) is expressed in the form

$$C_k = C_k(q, p), \qquad \psi_k = \omega_k(C)t + \varphi_k = \psi_k(q, p) \quad (k = 1, \dots, n), \tag{21}$$

then, using the result of V. M. Volosov [5], instead of (16) we obtain the equivalent system of equations

$$\frac{dC_k}{dt} = \sum_i Q_i \frac{\partial C_k(q,p)}{\partial p_i}, \qquad \frac{d\psi_k}{dt} = \omega_k(C) + \sum_i Q_i \frac{\partial \psi_k(q,p)}{\partial p_i} \quad (i,k = 1,\dots,n), \tag{22}$$

after which the averaging method can be applied to (22) also in the case of nonconservative systems with many degrees of freedom, analogously to how this was done above.

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* If system (2), moreover, is completely expressed ($m = n - 1$)—this is the case of the rotating phase, studied by N. N. Bogolyubov and D. N. Zubarev [4]. As applied to systems of a somewhat more general type than (1), for the case of single-frequency oscillations (when $k \neq 1$, all $\nu_k = \psi_k = 0$) and independence of the right-hand sides from the explicit time t, a similar consideration was carried out by V. M. Volosov [5].

Note: Figure translations are in progress. See original paper for figures.