Hydromechanics

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Abstract

Full Text

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On the Equations of Relaxation Hydrodynamics

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1. The time τ for the establishment of local thermodynamic equilibrium varies over wide limits for different processes. If, during the time τ, the hydrodynamic quantities manage to change substantially, it is necessary to take the relaxation process explicitly into account in the equations of hydrodynamics. Such conditions arise, for example, in supersonic flows, in particular when a medium passes through discontinuity surfaces, and also in the propagation of sound of sufficiently high frequencies.

2. The system of equations of "relaxation hydrodynamics" has properties substantially different from those of the equations of "ordinary" hydrodynamics. Some of these properties do not depend on the assumption concerning the order of magnitude of τ and do not pass into the properties of equilibrium hydrodynamics as τ tends to zero. This is the case, for example, with the characteristics of the system of equations for nonstationary one-dimensional motion without viscosity and heat conduction (1). Meanwhile, the equations of ordinary hydrodynamics prove to be quite suitable for sufficiently small τ. Therefore, along with the general properties of the equations of relaxation hydrodynamics, it is of interest to examine in more detail the transition to equilibrium hydrodynamics by means of a formal expansion of the solution in a series in powers of τ.

3. In what follows we assume that the nonequilibrium states under consideration admit a thermodynamic description. As is known, this possibility is connected with the very high rate at which the equilibrium distribution of energy over the translational degrees of freedom of gas particles is established. Let ξ be a parameter characterizing the degree of deviation of the state from complete local equilibrium. Its equilibrium value ξ_0 is determined by the equation $\varepsilon_\xi(\rho, s, \xi_0) = 0$, where $\varepsilon_\xi \equiv (\partial\varepsilon/\partial\xi)_{\rho,s}$; ε is the internal energy per unit mass, regarded as a function of the specific entropy s, the density ρ, and the parameter ξ.

To find $d\varepsilon$ during the motion of the medium, neglecting heat conduction and viscosity, we have $d\varepsilon = -p\, d(1/\rho)$. Hence, from the general expression for $d\varepsilon$:

$$d\varepsilon = -p\, d(1/\rho) + T ds + \varepsilon_\xi\, d\xi$$

we obtain

$$\frac{ds}{dt} = -\frac{1}{T}\varepsilon_\xi \frac{d\xi}{dt}, \tag{1}$$

where $d/dt \equiv \partial/\partial t + \mathbf{v}$ grad, $\mathbf{v}$ is the velocity of the medium. Choosing, as the "flux" and "force" (in the sense of the thermodynamics of irreversible processes), $d\xi/dt$ and $-\varepsilon_\xi$, respectively, one may write the kinetic equation in the form

$$\frac{d\xi}{dt} = -K\varepsilon_\xi(\rho, s, \xi), \tag{2}$$

where K is a phenomenological coefficient which is, generally speaking, a function of state. Equations (1) and (2), together with the equations of continuity and of motion of an ideal fluid, form a closed system.

equations of relaxation hydrodynamics

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0; \tag{3}$$

$$\rho \frac{d\mathbf{v}}{dt} + \operatorname{grad} p = 0; \tag{3'}$$

$$\frac{ds}{dt} = \frac{K}{T}(\varepsilon_\xi)^2; \tag{3''}$$

$$\frac{d\xi}{dt} = -K\varepsilon_\xi. \tag{3'''}$$

Here the pressure p and temperature T are determined by the expressions $p = \rho^2\varepsilon_\rho$, $T = \varepsilon_s$. From (2) it is seen that the coefficient K is related to the relaxation time τ of small deviations from local thermodynamic equilibrium by

$$\tau = \frac{1}{K\varepsilon_{\xi\xi}} \tag{4}$$

(assuming in this case that K does not depend on $(\xi - \xi_0)$ and that the change in entropy is a quantity of second order of smallness).

4. Let us consider some properties of the system (3). Taking rot of (3′), after some transformations we obtain

$$\frac{\partial \mathbf{R}}{\partial t} + \operatorname{rot}[\mathbf{R} \times \mathbf{v}] = \frac{p_s}{\rho^2}[\operatorname{grad} p \times \operatorname{grad} s] + \frac{p_\xi}{\rho^2}[\operatorname{grad} p \times \operatorname{grad} \xi], \tag{5}$$

where $\mathbf{R} \equiv \operatorname{rot} \mathbf{v}$. Thus, the vorticity of the velocity changes with time. However, for small deviations from equilibrium, Helmholtz' s theorem on conservation of vorticity holds up to terms quadratic in the amplitude of the deviation. For small perturbations propagating in a quiescent homogeneous equilibrium medium, linearization of equations (3) leads to the following equation, which is a generalization of the law of propagation of small perturbations in a relaxing medium:

$$\tau \frac{\partial}{\partial t}\left(\frac{\partial^2 \mathbf{v}}{\partial t^2} - c_\infty^2 \Delta \mathbf{v}\right) + \frac{\partial^2 \mathbf{v}}{\partial t^2} - c_0^2 \Delta \mathbf{v} = 0, \tag{6}$$

where $c_\infty^2 = p_\rho$, $c_0^2 = c_\infty^2 - p_\xi \varepsilon_{\xi\rho}/\varepsilon_{\xi\xi}$. From the equilibrium condition $\varepsilon_\xi = 0$ it is seen that c_0^2 is the derivative of pressure with respect to density under the condition that equilibrium is preserved. c_∞ and c_0 are the speeds of sound propagation at infinitely high and infinitely low frequencies, respectively, as follows from considering solutions of the type $\exp\{i(\mathbf{kr} - \omega t)\}$ [2]. Equations analogous to (6) can also be obtained for other hydrodynamic quantities. In this approximation the entropy is conserved, and the motion may be regarded as potential. For $\tau \to \infty$ or $\tau \to 0$, (6) passes into the ordinary wave equations, but with different propagation speeds, respectively c_∞ and c_0.

The characteristics of the system (3) in the case, for example, of one-dimensional nonstationary motion are determined by the speed c_∞ of high-frequency sound [1]. For a stationary flow along the x-axis with velocity u, the tangent of the angle of the characteristic with the x-axis is equal to $1/\sqrt{M_\infty^2 - 1}$, where $M_\infty = u/c_\infty$. Denoting by η the tangent of the angle of inclination of the characteristic to the t-axis, we obtain, by the general rules,

$$\eta = v \tag{7}$$

or

$$\eta = v \pm c_\infty. \tag{7'}$$

In addition to equations (3″) and (3‴), which already have characteristic form, the following two equations of relaxation hydro-

dynamics in characteristic form:

$$c_\infty^2 \frac{D_\pm \rho}{Dt} \mp \rho c_\infty \frac{D_\pm v}{Dt} + p_s \frac{D_\pm s}{Dt} + p_\xi \frac{D_\pm \xi}{Dt} = \pm K \varepsilon_\xi \left(\frac{\varepsilon_\xi}{T} p_s - p_\xi\right), \tag{8}$$

where

$$\frac{D_\pm}{Dt} \equiv \frac{\partial}{\partial t} + (v \pm c_\infty)\frac{\partial}{\partial x} \equiv \frac{d}{dt} \pm c_\infty \frac{\partial}{\partial x}. \tag{9}$$

5. It is interesting that this last result, like (7′), does not depend on the rate of the relaxation processes, i.e., it remains valid also in the case when the rate of the latter is so large that, in practice, one may use the equations of equilibrium hydrodynamics. Thus, from the standpoint of relaxation hydrodynamics, the Mach lines observed experimentally (for small τ) (along which small disturbances propagate) do not coincide with the characteristics. In this connection two questions arise: 1) the role of the directions determined by the velocities c_0 and c_∞ in flows with arbitrary τ ([2,3]), and 2) the question considered below concerning the formal passage to the limiting case of small τ.

6. Let $\mu = v\tau/L \ll 1$, where L is the distance over which the hydrodynamic quantities change substantially.

If one seeks the solution of system (3) in the form of a series

$$y(\mathbf{r},t) = y^{(0)}(\mathbf{r},t) + \mu y^{(1)}(\mathbf{r},t) + \cdots, \tag{10}$$

where $y(\mathbf{r},t)$ is any of the hydrodynamic quantities, then the following result is obtained: the equation of the first approximation for $\xi^{(0)}$ describes the process of relaxation of the parameter ξ with characteristic time τ; quantities such as $\rho^{(0)}$ remain constant in this approximation. In the second approximation expressions divergent with time appear. Therefore, this method of expansion in μ is suitable for describing the motion only over sufficiently small intervals of time. We note that the characteristics of the equations obtained in this way are determined by the velocity c_∞.

The transition to the equation of ordinary hydrodynamics can be carried out by means of the following expansion for ξ:

$$\xi = \xi^{(0)}(\mathbf{r};\rho,s,\mathbf{v}) + \mu\xi^{(1)}(\mathbf{r};\rho,s,\mathbf{v}) + \cdots, \tag{11}$$

where $\xi^{(0)} = \xi_0(\rho,s)$, and the coefficients of μ^n are functionals of the fields $\rho, s, \mathbf{v}$. This means that, in the first approximation, the relaxation process of ξ is regarded as completed, and the deviations of ξ from the equilibrium value are determined at each given instant of time by the fields $\rho, s, \mathbf{v}$. Under these conditions, evidently, the following expansions are valid for the time derivatives of the hydrodynamic quantities:

$$\frac{\partial\rho}{\partial t} = \mu R^{(1)}(\mathbf{r};\rho,s,\mathbf{v}) + \mu^2 R^{(2)}(\mathbf{r};\rho,s,\mathbf{v}) + \cdots; \tag{12'}$$

$$\frac{\partial\mathbf{v}}{\partial t} = \mu \mathbf{V}^{(1)}(\mathbf{r};\rho,s,\mathbf{v}) + \mu^2 \mathbf{V}^{(2)}(\mathbf{r};\rho,s,\mathbf{v}) + \cdots; \tag{12''}$$

$$\frac{\partial s}{\partial t} = \mu S^{(1)}(\mathbf{r};\rho,s,\mathbf{v}) + \mu^2 S^{(2)}(\mathbf{r};\rho,s,\mathbf{v}) + \cdots, \qquad (12"')$$

where the coefficients of μ^n are also functionals of $\rho(\mathbf{r}), s(\mathbf{r}), q(\mathbf{r})$. The expansions (11), (12) are essentially quite analogous to methods developed in nonlinear mechanics and statistical physics (see, for example, ([6])) in connection with the appearance of secular terms in expansions of the type (10). Comparing (12) and (3) and taking into account that derivatives of the n-th order with respect to the coordinates of the hydrodynamic quantities are quantities of order μ^n, we find

$$R^{(1)} = -\mathbf{v}\,\mathrm{grad}\,\rho - \rho\,\mathrm{div}\,\mathbf{v}; \qquad R^{(2)} = R^{(3)} = \cdots = 0. \qquad (13)$$

From (12′) and (3′), the expansion for p is

$$p = p_0(\rho,s) + \mu\left(\frac{\partial p}{\partial \xi}\right)_{\xi=\xi_0} \xi^{(1)} + \cdots, \qquad (14)$$

where $p_0(\rho,s)$ is the equilibrium pressure; we find

$$\mathbf{v}^{(1)} = -(\mathbf{v}\nabla)\mathbf{v} - \frac{1}{\rho}\,\mathrm{grad}\,p_0; \qquad \mathbf{v}^{(2)} = -\frac{1}{\rho}\,\mathrm{grad}\left(p_\xi \xi^{(1)}\right). \qquad (15)$$

Similarly, $S^{(1)}, S^{(2)}$, etc., are determined.

Keeping in (12′) the terms of first order in μ, we obtain the equation of motion of an ideal fluid with the equilibrium value of the pressure. (We regard μ as a formally introduced parameter, putting $\mu = 1$ in the final results.)

7. To obtain the equation of motion in the next approximation, it is necessary to find $\xi^{(1)}$ in expression (12) for $\mathbf{v}^{(2)}$.

On the basis of (11), (12), (12′), (12″), (13), and taking into account that ds/dt is a quantity of second order of smallness, (3″) can, to terms of order μ^2, be represented in the form

$$\frac{\partial \xi_0}{\partial \rho}\rho\ \mathrm{div}\,\mathbf{v} = \frac{1}{\tau}\xi^{(1)}. \qquad (16)$$

Substituting this value of $\xi^{(1)}$ into (15) and (12′), we obtain the equation of motion in the second approximation

$$\rho\frac{d\mathbf{v}}{dt} = -\,\mathrm{grad}\,p_0 + \mathrm{grad}\left(\zeta\ \mathrm{div}\,\mathbf{v}\right), \qquad (17)$$

where

$$\zeta = -\left(\frac{\partial p}{\partial \xi}\right)_{\xi=\xi_0} \tau \frac{\partial \xi_0}{\partial \rho} \rho \tag{18}$$

or

$$\zeta = \tau \rho \left(c_\infty^2 - c_0^2\right) \tag{18'}$$

has the meaning of the second coefficient of viscosity.

The derivation presented, under the assumption $\mu \ll 1$, is not, however, restricted by the condition of small variations of the hydrodynamic quantities and is therefore also valid outside the acoustic approximation (cf. ([2], [5])). In the same way, equations can evidently also be obtained in higher approximations.

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Note: Figure translations are in progress. See original paper for figures.