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THE PROBLEM OF THE DECAY OF AN ARBITRARY DISCONTINUITY FOR A SYSTEM OF FIRST-ORDER QUASILINEAR EQUATIONS

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In the present article we consider the problem of the decay of an arbitrary discontinuity for a hyperbolic system of two quasilinear equations of the form (1). The existence and uniqueness of a self-similar solution of this problem are proved for a certain class of systems (1). This class includes, in particular, the equations of gas dynamics.

In the general setting, questions of existence and uniqueness of a generalized solution of the Cauchy problem for systems of the form (1) have so far scarcely been discussed in the literature. We point to the work (^{6}), where uniqueness of a piecewise smooth solution was proved for systems of a somewhat narrower class than the one considered here. The problem of the decay of a discontinuity was studied in (^{4}), where the existence of its self-similar solution was proved for the case of a sufficiently weak discontinuity. Some remarks concerning the problem of the decay of a discontinuity are contained in (^{5}).

1. We consider the hyperbolic system of quasilinear equations

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u)}{\partial x} = 0 \quad (i = 1, 2). \tag{1}$$

The assumption of hyperbolicity means that the eigenvalues $\xi_1(u)$ and $\xi_2(u)$ of the matrix $(\partial\varphi_i/\partial u_j)$ are real and distinct in some domain of variation of u. For simplicity, let this domain be the entire plane (u_1, u_2). We assume the functions $\varphi_i(u)$ to be continuous and twice continuously differentiable in this plane. Let

$$\xi_1(u) > \xi_2(u). \tag{2}$$

The Cauchy problem for system (1) with initial data prescribed on the line $t = 0$ is called the problem of the decay of a discontinuity for system (1) if the initial data have the form:

$$u_j(0,x) = \begin{cases} u_j^-, & x < 0, \\ u_j^+, & x > 0, \end{cases} \tag{3}$$

where $u_j^+,\ u_j^-$ are arbitrary constants.

By a solution of problem (1)–(3) we mean, as usual, a generalized solution [5], and in our consideration we restrict ourselves only to stable generalized solutions. In this connection, we call a certain solution $u_i(t,x)$ of problem (1) –(3) stable if every line of discontinuity $x = x(t)$ in this solution satisfies, at points of smoothness, one of the following two conditions (we denote $D = x'(t)$):

$$\xi_1(u(t,x+0)) \le D \le \xi_1(u(t,x-0)), \quad D > \xi_2(u(t,x-0)); \tag{4}$$

$$\xi_2(u(t,x+0)) \le D \le \xi_2(u(t,x-0)), \quad D < \xi_1(u(t,x+0)). \tag{5}$$

The problem (1)–(3) is invariant with respect to the similarity transformation $t \to kt,\ x \to kx$, so that its solution is naturally sought in the form $u_i = u_i(x/t)$, called a self-similar solution.

We shall prove that, for a certain class of systems (1), the problem (1)–(3) has a unique self-similar solution.

Definition. We shall say that system (1) belongs to the class V if it satisfies the following requirements:

A. For every pair of points u', u'' there exists at least one point $\tilde{u}$ such that

$$\varphi_i(u'') - \varphi_i(u') = \sum_{j=1}^{2} \frac{\partial \varphi_i}{\partial u_j}(\tilde{u})(u_j'' - u_j') \qquad (i = 1, 2),$$

$$\tilde{u} \to u' \quad \text{as } u'' \to u'.$$

B. For every pair of points u'' and u'

$$\xi_1(u'') - \xi_2(u') \ge \varepsilon > 0, \qquad \Delta_{12}(u', u'') = \begin{vmatrix} c_{11}(u') & c_{12}(u') \\ c_{21}(u'') & c_{22}(u'') \end{vmatrix} \ne 0,$$

$c_j(u) = (c_{j1}(u),\ c_{j2}(u))$ is the normalized eigenvector of the matrix $(\partial \varphi_i / \partial u_j)$ corresponding to the eigenvalue ξ_j.

Let us note that the property of system (1) of belonging to the class V is not invariant with respect to a transformation of the independent variables depending on the solution. Below we shall use this circumstance to extend the class of systems under consideration.

2. We restrict ourselves to considering systems of class V both of whose characteristics are essentially noncontact, i.e. cannot be discontinuity lines of the solution. A sufficient condition for noncontactness of the characteristics is that the derivatives $\partial\xi_k/\partial v_k$ do not vanish, where v_k is the Riemann invariant corresponding to the characteristic value ξ_k (^{2,3}). In another form this condition can be written as follows (^{1}):

$$c_k \operatorname{grad} \xi_k = \sum_{j=1}^{2} c_{kj} \frac{\partial \xi_k}{\partial u_j} \neq 0. \tag{6}$$

The system of equations of hydrodynamics for the case of barotropic flow of a compressible gas in the Lagrangian representation belongs to the class V, if it is considered in the domain $\sqrt{-p'(\eta)} \geq \varepsilon > 0$ (p, η—pressure and specific volume). Condition (6) then means the constant sign of $p''(\eta)$. The class V also includes the system of equations of magnetohydrodynamics describing barotropic flow of an ideally conducting gas in a transverse magnetic field. Condition (6) for this system gives

$$p''(\eta) + \frac{3H^2}{4\pi\eta^4} > 0.$$

Theorem. *The problem of the decay of an arbitrary discontinuity for system (1), belonging to the class V and satisfying condition (6), has a unique self-similar solution.*

We give an outline of the proof. We shall carry out the consideration in the plane of the invariants (v_1, v_2).

Discontinuity lines satisfying the stability conditions (4) and (5) will be called shock waves of the 1st and 2nd types, respectively. It is proved that the systems under consideration allow in a solution only essentially noncontact shock waves, i.e. such discontinuity lines whose direction does not coincide with the characteristic either to the left or to the right of the point of discontinuity.

By a rarefaction wave of the j-th type we mean a region of the x/t axis in which $\xi_j(v) = x/t$. The points with which a given point can be connected by a rarefaction wave—

rarefaction waves of the 1st or 2nd type are situated on the straight lines $v_1 =$ const and $v_2 =$ const (Fig. 1). The direction of the arrow corresponds to motion along the x/t axis from the left state to the right one.

The points with which the given point v_0 can be joined by a shock wave are found from the equation

$$(\varphi_1 - \varphi_{10})(u_2 - u_{20}) - (\varphi_2 - \varphi_{20})(u_1 - u_{10}) = 0$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

and the stability conditions (4)–(5). It is proved that they are situated on lines called shock adiabats (Fig. 2) and possessing the following basic properties: 1) the shock adiabats of the 1st and 2nd type corresponding to the point v_0 intersect only at this point; 2) the shock adiabat of the j-th type intersects every straight line $v_j = \text{const}$ at one and only one point; 3) points lying on the adiabat of the j-th type can be connected with the point v_0 by a stable shock wave of the j-th type (the direction of the arrow corresponds to motion along the x/t axis from the left state to the right one); 4) there are no points that are connected with v_0 by a shock wave and do not lie on the shock adiabats corresponding to the point v_0.

Fig. 1

Fig. 2

From the stability conditions it follows that the self-similar solution being constructed must contain no more than one wave (shock or rarefaction) of each type, and the wave of the 1st type must be situated on the x/t axis to the right of the wave of the 2nd type. Consequently, the theorem reduces to the assertion that, for any v^- and v^+, there exists one and only one point v^* (Fig. 3) such that the curve L_1, corresponding to this point, passes through the point v^+. The validity of this assertion is established without difficulty.

Fig. 3

Let us note that in the case when one characteristic or both are contact ones, i.e. there exist regions in which

$$c_j \operatorname{grad} \xi_j = 0,$$

the theorem is also valid.

The theorem is also valid for systems (1) that do not belong to class V, but can be reduced to systems of this class by means of a nonsingular transformation of the independent variables, depending on the solution, described, for example,

Fig. 3

Figure 3: Fig. 3

in work (3). An example of such a transformation is the passage from Eulerian coordinates to Lagrangian ones in hydrodynamics.

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