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Abstract

Full Text

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On Conditions for Uniqueness of the Solution of an Abstract Cauchy Problem

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The abstract Cauchy problem, in the terminology of Hille ([1]), consists in finding a vector function $x(t)$ $(t \geq 0)$ in a Banach space satisfying the equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t > 0) \tag{1}$$

and the initial condition

$$x(0) = x_0. \tag{2}$$

Here A is a given linear operator, x_0 a given vector.

A vector function $x(t)$ $(t \geq 0)$ will be called a **weak solution** of problem (1)–(2) if it: 1) is weakly absolutely continuous and weakly differentiable almost everywhere on the half-axis $t > 0$; 2) satisfies equation (1) almost everywhere; 3) is weakly continuous at $t = 0$; 4) satisfies the initial condition (2). Replacing in this definition the word "weak" by the word "strong," we obtain the definition of a **strong solution**.

Theorem 1. *If on some ray L of the positive half-plane the spectrum of the operator A is absent, and if the resolvent R_λ of the operator A on the ray L is a function of finite degree, i.e.*

$$\sigma \equiv \overline{\lim_{\lambda \to +\infty}} \frac{\ln \|R_\lambda\|}{\lambda} < \infty, \tag{3}$$

then the Cauchy problem (1)–(2) cannot have two different weak solutions.

Proof. Let $x(t)$ be a weak solution of the problem

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t > 0),$$

$$x(0) = 0.$$

We shall show that

$$x(t) = 0 \qquad (0 \le t < \infty). \tag{4}$$

Take an arbitrary linear functional f and form the function

$$\varphi(t,\lambda) = f[R_\lambda x(t)] \qquad (t \ge 0,\ \lambda \in L). \tag{5}$$

This function on the half-axis $t > 0$ is an absolutely continuous solution of the equation

$$\frac{\partial \varphi(t,\lambda)}{\partial t} - \lambda\varphi(t,\lambda) = f[x(t)],$$

since

$$R_\lambda A \subset \lambda R_\lambda + E.$$

Moreover, $\varphi(t,\lambda)$ is continuous at $t = 0$ and $\varphi(0,\lambda) = 0$. Consequently,

$$\varphi(t,\lambda) = \int_0^t f[x(t-\tau)]e^{\lambda\tau}\,d\tau. \tag{6}$$

From formula (6) it is clear that, for fixed t, $\varphi(t,\lambda)$ is an entire function of λ of finite degree. Applying the well-known theorem of Pólya* ([2]), it is easy to show that the indicator diagram of the function $\varphi(t,\lambda)$ is the smallest interval of the real axis containing all those values τ, $0 \leqslant \tau \leqslant t$, for which $f[x(t-\tau)] \neq 0$. However, in view of (5) and (3), the indicator diagram of the function $\varphi(t,\lambda)$ lies in the half-plane $\operatorname{Re}\tau \leqslant \sigma$. Therefore, for $t > \sigma$,

$$f[x(t-\tau)] = 0 \qquad (\sigma \leqslant \tau \leqslant t),$$

i.e.

$$f[x(\tau)] = 0 \qquad (0 \leqslant \tau \leqslant t-\sigma).$$

By virtue of the arbitrariness of $t > \sigma$ and of the linear functional f, identity (4) holds. The theorem is proved.

The theorem proved is sharp in the following sense.

Theorem 2. *Let $\rho(\lambda)$ $(\lambda > 0)$ be a positive continuous function increasing as $\lambda \to +\infty$ faster than any exponential, i.e. such that*

$$\frac{\ln \rho(\lambda)}{\lambda} \longrightarrow +\infty.$$

Then there exists a Hilbert space and, in it, a linear operator A such that:

a) the positive half-axis contains no spectrum of the operator A, and

$$\|R_\lambda\| \leqslant \rho(\lambda) \tag{7}$$

for sufficiently large $\lambda > 0$;

b) the Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t > 0), \tag{8}$$

$$x(0) = 0 \tag{9}$$

has a nontrivial strong solution.

Proof. As the operator A we take the differentiation operator $-d/ds$ in the space $\mathcal{L}^2_\alpha(0,\infty)$ of functions square-summable on the half-axis $[0,\infty)$ with some positive measurable weight $\alpha(s)$. The domain of definition of this operator consists of all absolutely continuous functions $X(s) \in \mathcal{L}^2_\alpha(0,\infty)$ whose derivative also belongs to $\mathcal{L}^2_\alpha(0,\infty)$.

In order to obtain a nontrivial strong solution of problem (8)–(9) for the chosen operator A, it is enough to take any finite continuously differentiable function $X(\tau)$ $(-\infty < \tau < \infty)$, equal to zero on the negative half-axis, and put $x(t) = X(t-s)$.

It remains to choose the weight so as to satisfy condition a). Put

$$\beta(\tau) = \sup_{s \geqslant 0} \frac{\alpha(s)}{\alpha(s+\tau)} \qquad (\tau \geqslant 0)$$

* See, for example, ([3]), pp. 113-116.

and require that for any $k > 0$ the inequality

$$\beta(\tau) \leq C(k) e^{-k\tau} \tag{10}$$

hold with some constant $C(k)$. We shall show that then, for $\lambda > 0$, the equation

$$AX - \lambda X = Y \tag{11}$$

is uniquely solvable for any $Y \in \mathcal{L}^2_\alpha(0,\infty)$.

The uniqueness of the solution follows from the fact that $e^{-\lambda s} \notin \mathcal{L}^2_\alpha(0,\infty)$, which in turn follows from the inequality

$$\alpha(s) \geq \frac{\alpha(0)}{\beta(s)} \geq \frac{\alpha(0)}{C(2\lambda)} e^{2\lambda s}.$$

A solution of equation (11) is the function

$$X(s) = e^{-\lambda s} \int_s^\infty Y(\tau) e^{\lambda\tau}\, d\tau = \int_0^\infty Y(s+\tau) e^{\lambda\tau}\, d\tau. \tag{12}$$

That the function (12) belongs to $\mathcal{L}^2_\alpha(0,\infty)$ is seen from the estimate

$$|X(s)|^2 \leq \int_0^\infty |Y(s+\tau)|^2 e^{(2\lambda+1)\tau}\, d\tau,$$

by virtue of which

$$\int_0^\infty |X(s)|^2 \alpha(s)\, ds \leq \int_0^\infty \beta(\tau) e^{(2\lambda+1)\tau}\, d\tau \int_0^\infty |Y(s)|^2 \alpha(s)\, ds.$$

The last inequality also shows that, for the given operator A,

$$\|R_\lambda\| \leq \left[\int_0^\infty \beta(\tau) e^{(2\lambda+1)\tau}\, d\tau \right]^{1/2} \quad (\lambda > 0). \tag{13}$$

Relying on estimate (13), we construct the weight $\alpha(s)$ so as to ensure inequality (7). Of course, condition (10) must be observed in doing so. Put

$$M(\lambda) = \frac{(2\lambda+1)\rho^2(\lambda)}{e^{2\lambda+1}-1},$$

and let $\lambda_0 > 0$ be so large that $\min_{\lambda\geq\lambda_0} M(\lambda) > 1$. Next put $\beta_0 = 1$ and successively define the numbers $\beta_1, \beta_2, \ldots$ so that

$$0 < \beta_{N+1} < \min_{\lambda\geq\lambda_0} \left\{ M(\lambda) - \sum_{n=0}^{N} \beta_n e^{(2\lambda+1)n} \right\} e^{-(2\lambda+1)(N+1)} \tag{14}$$

and, moreover, so that

$$\beta_{N+1} < \frac{\beta_N^2}{\beta_{N-1}}, \qquad \beta_N < e^{-N^2} \quad (N = 1, 2, \ldots). \tag{15}$$

Introduce the step function $B(s)$, setting

$$B(s) = \beta_N \qquad (N \leq s < N+1;\ N = 0, 1, 2, \ldots).$$

and for the function $-\ln B(s)$ we construct a convex majorant $K(s)$ so that $K(0) = 0$. This is possible by virtue of (15).

Finally, put

$$\alpha(s) = e^{K(s)}.$$

Then

$$\frac{\alpha(s)}{\alpha(s+\tau)} = e^{K(s)-K(s+\tau)} \leqslant e^{-K(\tau)} \leqslant B(\tau).$$

Consequently, for the weight thus chosen we shall have

$$\beta(\tau) \leqslant B(\tau),$$

whence, according to (13), (14), it follows that

$$\|R_\lambda\| \leqslant \left[\sum_{n=0}^{\infty} \beta_n e^{(2\lambda+1)n} \cdot \frac{e^{2\lambda+1}-1}{2\lambda+1}\right]^{1/2} \leqslant \rho(\lambda).$$

The theorem is proved.

In conclusion we note that uniqueness of the solution of the Cauchy problem may fail also because of the presence, in the right half-plane, of arbitrarily distant points of the spectrum of the operator A; an example is provided by the operator

$$-\frac{d}{ds} \quad \text{in } \mathcal{L}^2(0, \infty).$$

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[1] E. Hille, Proc. Int. Congr. of Math., Amst., 1954, 3, p. 109.
[2] G. Pólya, Math. Zs., 29, 549 (1929).
[3] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.