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## Abstract

## Full Text

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## PSEUDONORMAL CORRELATION AND ITS VARIOUS GENERALIZATIONS

*(Presented by Academician S. N. Bernstein on 16 I 1960)*

1. **Definition.** We shall call pseudonormal the correlation dependence specified in the plane $-\infty < x,\ y < \infty$ by the joint probability density

$$P_{-1/2,\lambda}(x,y) = \frac{1}{4\pi\sqrt{1-\lambda^2}} \exp\left[-\frac{x^2+y^2-2\lambda xy}{2(1-\lambda^2)}\right] + \frac{1}{4\pi\sqrt{1-\lambda^2}} \exp\left[-\frac{x^2+y^2+2\lambda xy}{2(1-\lambda^2)}\right], \tag{1}$$

where $|\lambda| < 1$.

Each of the terms in (1) is a normal correlation density multiplied by 1/2, with correlation coefficient $\pm\lambda$, respectively; if each of these densities is replaced by the bilinear expansion in Hermite polynomials ((1), p. 399), then for (1) we obtain an expansion containing only polynomials of even order

$$P_{-1/2,\lambda}(x,y) = \frac{\exp\left[-\dfrac{x^2+y^2}{2}\right]}{2\pi} \left[1 + \sum_{k=1}^{\infty} H_{2k}(x)H_{2k}(y)\lambda^{2k}\right] = f(x^2,y^2). \tag{2}$$

For the correlation (1), as for normal correlation, the a priori densities $p(x)$, $p(y)$ are normal:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad p(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}. \tag{3}$$

However, the correlation coefficient and the correlation ratio are equal to zero, as are all conditional moments of odd order.

The maximal correlation coefficient $(^2)$ is $R^* = \lambda^2$; as $\lambda \to 0$, the dependence between $x$ and $y$ rapidly weakens.

**Remark.** It is interesting to note that the sum of the series

$$\psi(x^2,y^2) = \frac{\exp\left[-\dfrac{x^2+y^2}{2}\right]}{2\pi} \left[1 + \sum_{k=1}^{\infty} H_{4k}(x)H_{4k}(y)\lambda^{4k}\right] \tag{4}$$

is a sign-changing function and is not a probability distribution density.

2. The density (1) is a particular case of a more general density, also distributed over the whole real plane:

$$P_{\alpha,\lambda}(x,y) = \frac{|xy|^{2\alpha+1}\exp\left[-\frac{x^2+y^2}{2(1-\lambda^2)}\right]}{(1-\lambda^2)^{\alpha+1}2^{2\alpha+2}\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{\left[\frac{xy\lambda}{2(1-\lambda^2)}\right]^{2k}}{k!\,\Gamma(\alpha+k+1)} =$$

$$= \frac{|xy|^{1+\alpha}\exp\left[-\frac{x^2+y^2}{2(1-\lambda^2)}\right]}{\Gamma(\alpha+1)|\lambda|^{\alpha}2^{2+\alpha}(1-\lambda^2)} I_{\alpha}\left(\frac{xy\lambda}{1-\lambda^2}\right), \tag{5}$$

where $I_\alpha(z)$ is a Bessel function of imaginary argument, $\alpha > -1$, $\lambda^2 < 1$. For $\alpha = -1/2$, (5) yields the density (1). The density (5) is of considerable interest in the theory of stationary Markov processes, and therefore we shall examine it in greater detail.

3. Multiplying (5) by 4, we obtain a density defining a correlation in the first quadrant; replacing $x$ and $y$, respectively, by $\sqrt{2x}$ and $\sqrt{2y}$, we reduce it to the form

$$F_{\alpha,\lambda}(x,y) = \frac{x^{\alpha}y^{\alpha}\exp\left[-\frac{x+y}{1-\lambda^2}\right]}{(1-\lambda^2)^{1+\alpha}\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{\left[\frac{xy\lambda^2}{(1-\lambda^2)^2}\right]^{k}}{k!\,\Gamma(\alpha+k+1)} =$$

$$= \frac{x^{\alpha}e^{-x}}{\Gamma(\alpha+1)} \frac{y^{\alpha}e^{-y}}{\Gamma(\alpha+1)} \left[1 + \sum_{k=1}^{\infty} L_k^{\alpha}(x) L_k^{\alpha}(y) \lambda^{2k}\right] \tag{6}$$

$$(0 \le x, y < \infty,\ \alpha > -1,\ \lambda^2 < 1),$$

where $L_k^{\alpha}(x)$ are the generalized Laguerre polynomials, orthogonal and normalized with weight

$$p(x) = \frac{x^{\alpha}e^{-x}}{\Gamma(\alpha+1)} \quad (0 \le x < \infty). \tag{7}$$

Formula (6) is an obvious consequence of the expansion

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} \widetilde{L}_n^{\alpha}(x) \widetilde{L}_n^{\alpha}(y) z^n =$$

$$= \frac{\exp\left[-\frac{z(x+y)}{1-z}\right]}{1-z}(zxy)^{-\alpha/2} I_\alpha\left(2\frac{\sqrt{zxy}}{1-z}\right), \quad |z|<1 \tag{8}$$

$((^3),$ p. $189),$

where

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x)\sqrt{\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}}, \quad n = 1, 2, ... \tag{9}$$

The correlation (6) is linear; the correlation coefficient is nonnegative, $R = \lambda^2$. For $\lambda = 0$, $x$ and $y$ are independent and are distributed according to the prior laws with density (7).

The characteristic function of the distribution (6) has the form

$$\varphi(t_1, t_2) = \mathbf{E}\exp[it_1x + it_2y] = \frac{1}{[1 - it_1 - it_2 - (1-\lambda^2)t_1t_2]^{1+\alpha}}. \tag{10}$$

For $\alpha = 0$, the distribution takes the simplest form, corresponding to the classical Laguerre polynomials $L_n(x)$, orthogonal with weight $e^{-x}$ on the half-line $0 \le x < \infty$.

4. For the prior distribution (7), the first four moments have the values:

$$m_1 = 1+\alpha; \quad m_2 = (1+\alpha)(2+\alpha); \quad \sigma^2 = 1+\alpha; \tag{11}$$

$$m_3 = (1+\alpha)(2+\alpha)(3+\alpha); \quad m_4 = (1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha).$$

After the linear change of variables

$$x = \sqrt{1+\alpha}\,x_1 + 1 + \alpha, \qquad y = \sqrt{1+\alpha}\,y_1 + 1 + \alpha$$

we obtain new centered and normalized variables $x_1, y_1$, connected by correlation dependence in the quadrant $-\sqrt{1+\alpha} \le x_1; y_1 < \infty$.

Let us note that the coefficient of asymmetry of the a priori distribution, according to (11), is equal to

$$c = \frac{m_3 - 3m_1m_2 + 2m_1^3}{\sigma^3} = \frac{2}{\sqrt{1+\alpha}}. \tag{12}$$

Expressing $\alpha$ through $c$, for the density and characteristic function of the joint distribution of $x_1$ and $y_1$ we obtain the expressions

$$F_c(x_1, y_1) = \frac{4}{c^2}\, F_{\frac{4}{c^2}-1,\lambda} \left( \frac{2}{c} x_1 + \frac{4}{c^2}; \frac{2}{c} y_1 + \frac{4}{c^2} \right), \tag{13}$$

$$\varphi_c(t_1, t_2) = \frac{\exp\left[-\frac{2i}{c}(t_1 + t_2)\right]}{\left[1 - \frac{c}{2} i(t_1 + t_2) - \frac{c^2}{4}(1 - \lambda^2) t_1 t_2\right]^{4/c^2}}, \tag{14}$$

where

$$-\frac{2}{c} \le x_1; y_1 < \infty, \qquad c > 0. \tag{15}$$

For $c = 2$ (when $\alpha = 0$), i.e. in the case of the classical Laguerre polynomials, all formulas are substantially simplified.

5. Let now $c \to 0$, i.e. let the asymmetry of the distribution decrease and the quadrant (15) expand, covering, in the limit, the entire plane. Then, as is easy to verify, the correlation (13), (14) approaches a normal one; namely, in the limit one obtains the normal correlation

$$\lim_{c \to 0} \varphi_c(t_1, t_2) = \exp\left[-\frac{1}{2}\left(t_1^2 + t_2^2 + 2\lambda^2 t_1 t_2\right)\right], \tag{16}$$

$$F_0(x_1, y_1) = \frac{1}{2\pi(1 - \lambda^4)} \exp\left[-\frac{x_1^2 + y_1^2 - 2\lambda^2 x_1 y_1}{2(1 - \lambda^4)}\right] \tag{17}$$

with positive correlation coefficient $\lambda^2$.

6. Let us note that the characteristic function of the conditional distribution of $y$ for a given $x$, with density $F_{\alpha,\lambda}(x, y)/p(x)$, where $F_{\alpha,\lambda}(x, y)$ is defined by formula (6), and $p(x)$ by formula (7), has the expression

$$\varphi_x(t_2) = \frac{\exp\left[\dfrac{i\lambda^2 x t_2}{1 - (1 - \lambda^2) i t_2}\right]}{\left[1 - (1 - \lambda^2) i t_2\right]^{1+\alpha}}, \tag{18}$$

by means of which it is not difficult to find the conditional coefficients of asymmetry $S_\alpha$ and excess $E_\alpha$ of the quantity $y$ for a given $x$:

$$S_\alpha = \frac{2(1 + \alpha)(1 - \lambda^2)^3 + 6\lambda^2(1 - \lambda^2)^2 x}{\left[(1 + \alpha)(1 - \lambda^2)^2 + 2\lambda^2(1 - \lambda^2) x\right]^{3/2}}, \tag{19}$$

$$E_\alpha + 3 = \frac{3(1-\lambda^2)^4(1+\alpha)(3+\alpha) + 12\lambda^2(1-\lambda^2)^3(3+\alpha)x + 12\lambda^4(1-\lambda^2)^2x^2}{\left[(1+\alpha)(1-\lambda^2)^2 + 2\lambda^2(1-\lambda^2)x\right]^2}.$$

Letting $c = \dfrac{2}{\sqrt{1+\alpha}}$ tend to zero, we obtain

$$\lim_{\alpha\to\infty} S_\alpha = 0, \qquad \lim_{\alpha\to\infty} E_\alpha = 0, \tag{20}$$

as should be the case in the limit for normal correlation.

7. Finally, let us note that the correlation (6) or (13), constructed from generalized Laguerre polynomials, and its limiting case (17)—the normal correlation constructed from Hermite polynomials—turn out to exhaust the class of densities possessing the following properties:

a) the eigenfunctions of the symmetric kernel $F(x, y)/\sqrt{p(x)p(y)}$ form a complete system of polynomials orthogonal on an infinite interval (in particular, the correlation is linear);

b) by means of the density $F(x, y)$ one can define a continuous Markov stationary process with continuous time.

**Remark 1.** The pseudonormal correlation (1) and its generalization (5) do not possess property a), since the corresponding kernels are not closed, and all polynomials of odd order not containing even powers of $x$ are orthogonal to them.

**Remark 2.** If $\alpha \to -1$, or, what is the same, $c \to \infty$, then the correlation (6) degenerates: the entire distribution is concentrated on the coordinate axes, where the density has an infinite value; off the coordinate axes the density tends to zero.

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*Note: Figure translations are in progress. See original paper for figures.*

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