MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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SOME GEOMETRIC PROPERTIES OF NOMOGRAPHABLE EQUATIONS

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Consider a function

$$w = f(u, v), \tag{1}$$

satisfying the following conditions:

1) the function $f(u, v)$ is defined for all values u, v in some neighborhood g of the point u_0, v_0;
2) in g the function $f(u, v)$ and its derivatives up to and including the second order are continuous;
3)

$$\frac{\partial f(u, v)}{\partial v} \neq 0 \quad \text{for } u = u_0,\ v = v_0. \tag{2}$$

If, for the given function $w = f(u, v)$, one can find a determinant

$$\begin{vmatrix} A_1(u) & A_2(u) & A_3(u) \\ B_1(v) & B_2(v) & B_3(v) \\ C_1(w) & C_2(w) & C_3(w) \end{vmatrix} \neq 0, \tag{3}$$

satisfying the condition

$$\begin{vmatrix} A_1(u) & A_2(u) & A_3(u) \\ B_1(v) & B_2(v) & B_3(v) \\ C_1(f(u, v)) & C_2(f(u, v)) & C_3(f(u, v)) \end{vmatrix} \equiv 0, \tag{4}$$

then the function $w = f(u, v)$ is called **nomographable**. In this case, for equation (1) one can construct a nomogram of aligned points.

From (4) it follows that

$$C_1(f)\,p_{23}(u,v) + C_2(f)\,p_{31}(u,v) + C_3(f)\,p_{12}(u,v) \equiv 0, \tag{5}$$

where

$$p_{ik} = \begin{vmatrix} A_i(u) & A_k(u) \\ B_i(v) & B_k(v) \end{vmatrix}$$

and they satisfy the conditions

$$\begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial p_{23}}{\partial u} & \dfrac{\partial p_{31}}{\partial u} & \dfrac{\partial p_{12}}{\partial u} \\ \dfrac{\partial^2 p_{23}}{\partial u^2} & \dfrac{\partial^2 p_{31}}{\partial u^2} & \dfrac{\partial^2 p_{12}}{\partial u^2} \end{vmatrix} \equiv 0; \tag{6}$$

$$\begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial p_{23}}{\partial v} & \dfrac{\partial p_{31}}{\partial v} & \dfrac{\partial p_{12}}{\partial v} \\ \dfrac{\partial^2 p_{23}}{\partial v^2} & \dfrac{\partial^2 p_{31}}{\partial v^2} & \dfrac{\partial^2 p_{12}}{\partial v^2} \end{vmatrix} \equiv 0. \tag{7}$$

With the aid of the functions p_{ik}, the functions $A_i(u), B_k(v)$ are determined ([1,2]).

Differentiating identities (5) with respect to u, v, from the relation f'_u/f'_v we obtain

$$C_1(f)\left[\frac{\partial p_{23}}{\partial u} f'_v - \frac{\partial p_{23}}{\partial v} f'_u\right] + C_2(f)\left[\frac{\partial p_{31}}{\partial u} f'_v - \frac{\partial p_{31}}{\partial v} f'_u\right] + C_3(f)\left[\frac{\partial p_{12}}{\partial u} f'_v - \frac{\partial p_{12}}{\partial v} f'_u\right] \equiv 0. \tag{8}$$

From (5) and (8) it follows that

$$\frac{C_1(f)}{\begin{vmatrix} \dfrac{\partial p_{31}}{\partial u} & \dfrac{\partial p_{12}}{\partial u} \\ p_{31} & p_{12} \end{vmatrix} f'_v - \begin{vmatrix} \dfrac{\partial p_{31}}{\partial v} & \dfrac{\partial p_{12}}{\partial v} \\ p_{31} & p_{12} \end{vmatrix} f'_u} = \frac{C_2(f)}{\begin{vmatrix} \dfrac{\partial p_{12}}{\partial u} & \dfrac{\partial p_{23}}{\partial u} \\ p_{12} & p_{23} \end{vmatrix} f'_v - \begin{vmatrix} \dfrac{\partial p_{12}}{\partial v} & \dfrac{\partial p_{23}}{\partial v} \\ p_{12} & p_{23} \end{vmatrix} f'_u} =$$

$$= \frac{C_3(f)}{\begin{vmatrix} \dfrac{\partial p_{23}}{\partial u} & \dfrac{\partial p_{31}}{\partial u} \\ p_{23} & p_{31} \end{vmatrix} f'_v - \begin{vmatrix} \dfrac{\partial p_{23}}{\partial v} & \dfrac{\partial p_{31}}{\partial v} \\ p_{23} & p_{31} \end{vmatrix} f'_u}. \tag{9}$$

It is easy to show that the functions $P_{ik} = p_{ik}\lambda(u,v)$ satisfy conditions (6), (7). Define $\lambda(u,v)$ so that

$$P_{23}^2 + P_{31}^2 + P_{12}^2 = 1.$$

Then P_{ik} may be regarded as the coordinates of a point on a sphere whose radius is equal to 1 and whose center is at the origin of the coordinate system.

The vector $\mathbf{N} = \{P_{23}, P_{31}, P_{12}\}$ is the unit normal vector to the sphere, and the coefficients D, D', D'' of the second quadratic form of the sphere have the values

$$D = -E, \qquad D' = -F, \qquad D'' = -G,$$

where E, F, G are the coefficients of the first quadratic form of the sphere.

From (6), (7) it follows that

$$\frac{\partial^2 P_{ik}}{\partial u^2} = \lambda_1 \frac{\partial P_{ik}}{\partial u} + \mu_1 P_{ik} = \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} \frac{\partial P_{ik}}{\partial u} - E P_{ik};$$

$$\frac{\partial^2 P_{ik}}{\partial v^2} = \lambda_2 \frac{\partial P_{ik}}{\partial v} + \mu_2 P_{ik} = \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} \frac{\partial P_{ik}}{\partial v} - G P_{ik}.$$

Thus,

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = 0. \tag{10}$$

Hence it follows that the lines $u = \text{const}$, $v = \text{const}$ are geodesic lines on the sphere.

Since

$$\frac{C_1(f)}{C_3(f)} = \frac{\begin{vmatrix} \frac{\partial P_{31}}{\partial u} & \frac{\partial P_{12}}{\partial u} \\ P_{31} & P_{12} \end{vmatrix} f'_v - \begin{vmatrix} \frac{\partial P_{31}}{\partial v} & \frac{\partial P_{12}}{\partial v} \\ P_{31} & P_{12} \end{vmatrix} f'_u}{\begin{vmatrix} \frac{\partial P_{23}}{\partial u} & \frac{\partial P_{31}}{\partial u} \\ P_{23} & P_{31} \end{vmatrix} f'_v - \begin{vmatrix} \frac{\partial P_{23}}{\partial v} & \frac{\partial P_{31}}{\partial v} \\ P_{23} & P_{31} \end{vmatrix} f'_u} = \delta_1,$$

$$\frac{C_2(f)}{C_3(f)} = \frac{\begin{vmatrix} \frac{\partial P_{12}}{\partial u} & \frac{\partial P_{23}}{\partial u} \\ P_{12} & P_{23} \end{vmatrix} f'_v - \begin{vmatrix} \frac{\partial P_{12}}{\partial v} & \frac{\partial P_{23}}{\partial v} \\ P_{12} & P_{23} \end{vmatrix} f'_u}{\begin{vmatrix} \frac{\partial P_{23}}{\partial u} & \frac{\partial P_{31}}{\partial u} \\ P_{23} & P_{31} \end{vmatrix} f'_v - \begin{vmatrix} \frac{\partial P_{23}}{\partial v} & \frac{\partial P_{31}}{\partial v} \\ P_{23} & P_{31} \end{vmatrix} f'_u} = \delta_2$$

depends only on $f(u,v)$, then from the conditions

$$f'_v \frac{\partial}{\partial u}\delta_k - f'_u \frac{\partial \delta_k}{\partial v} = 0 \qquad (k=1,2)$$

we obtain Gronwall' s condition

$$M = \left(\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \right) f'_u + \left(\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \right) f'_v, \tag{11}$$

where

$$M = \frac{f''_{uu} {f'}_v^2 - 2 f''_{uv} f'_u f'_v + f''_{vv} {f'}_u^2}{f'_u f'_v}. \tag{12}$$

From condition (5) we find that the vector

$$\mathbf{C}(f) = \{C_1(f),\ C_2(f),\ C_3(f)\}$$

is situated in the tangent plane to the sphere.

At the points of the line $f(u,v) = c$ the vector $\mathbf{C}$ has a constant direction and is situated at each point of this line in the tangent plane to the sphere; consequently, the lines $f(u,v) = c$ are arcs of great circles or geodesic lines on the sphere.

On the basis of conditions (2), from the equation $f(u,v) = c$ we determine

$$v = \sigma(u,c), \qquad \frac{dv}{du} = -\frac{f'_u}{f'_v}$$

along the geodesics. Substituting the values $v = \sigma(u,c)$, $dv/du = -f'_u/f'_v$ into the differential equation of the geodesics, we obtain

$$M = \left(\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \right) f'_u + \left(\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \right) f'_v; \tag{13}$$

$$M = \frac{f''_{uu} {f'}_v^2 - 2 f''_{uv} f'_u f'_v + f''_{vv} {f'}_u^2}{f'_u f'_v} \tag{14}$$

for $v = \sigma(u,c)$.

Conditions (13), (14) are obtained from Gronwall' s conditions (11), (12) for $v = \sigma(u,c)$. But since for any values (u,v) one can determine c so that $v = \sigma(u,c)$, conditions (13), (14) will hold for any values (u,v).

Thus, Gronwall' s condition is the condition that any values $(u, v) \subset g$ and the derivative $dv/du = -f'_u/f'_v$, determined from the equation $f(u, v) = c$, satisfy the differential equation of the geodesics.

On the basis of the results obtained, one can construct a differential-geometric theory of nomographable equations.

The coefficients of the first quadratic form of the sphere satisfy Gronwall' s conditions (11), (12), the Gauss equation, and equations (10). The Peterson-Codazzi conditions are satisfied identically by virtue of conditions (10).

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Note: Figure translations are in progress. See original paper for figures.