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Abstract

Full Text

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L. N. GUTMAN and F. I. FRANKL

A THERMO-HYDRODYNAMIC MODEL OF THE BORA

(Presented by Academician A. A. Dorodnitsyn, 15 X 1959)

In this work an attempt is made to construct a thermo-hydrodynamic theory of the bora* on the basis of works on the theory of critical flows in a Laval nozzle and in a spillway ([2,3]), as well as works on the flow of air past mountains ([4,5]).**

Let us consider, in a rectangular coordinate system (x, y, z) (the x, y axes are directed horizontally, z upward), the steady nonlinear problem of the overflow of a cold air mass across an infinitely long mountain of cylindrical form. We direct the y axis along the generatrix of the mountain surface. Then the equation of this surface can be written in the form $z = \delta(x)$, where $\delta(x)$ will be regarded as a prescribed function, sufficiently smooth that vertical accelerations may be neglected ([7]).

We place the origin of coordinates near the base of the mountain on the windward side and shall assume the velocity U of the undisturbed flow (at $x = -\infty$) to be independent of the coordinates and directed along the x axis, i.e., perpendicular to the ridge. Neglecting the Coriolis force, we confine ourselves to a plane problem, assuming that nothing depends on y. We shall proceed from the general equations of thermo-hydrodynamics of the atmosphere, simplified by the theory of convection ([8]) and by neglecting internal viscosity and turbulence. Let η be the height of the surface (above the base of the mountain) separating the flow of cold air from the warmer air situated above, which we assume to be motionless. In contrast to all preceding works, we shall assume that η is a function of x to be determined.

After all simplifications we arrive at the following initial system of equations:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -R\theta\frac{\partial}{\partial x}\left(\frac{p'}{P}\right); \qquad R\theta\frac{\partial}{\partial z}\left(\frac{p'}{P}\right) = \lambda\vartheta \quad \left(\lambda = \frac{g}{\theta}\right);$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0; \qquad u\frac{\partial \vartheta}{\partial x} + w\frac{\partial \vartheta}{\partial z} + \mu w = 0 \quad (\mu = \gamma_a - \gamma). \tag{1}$$

Here u, w are the components of the wind velocity along the x, z axes, respectively; R is the gas constant for air; g is the acceleration due to gravity; γ is the

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vertical temperature gradient in the undisturbed flow; γ_a is the dry-adiabatic gradient; ϑ, p' are the deviations of temperature and pressure from $\theta(z)$ and $P(z)$, the values of these elements at $x = -\infty$. Of course, $\theta(z)$ and $P(z)$ must satisfy the equation of statics

$$R\theta \frac{dP}{dz} = -Pg \tag{2}$$

* A detailed description and study of the phenomenon, as well as an account of the history of the development of our knowledge about it, can be found in the fundamental monograph ([1]) devoted to the Novorossiisk bora. Let us note in passing that such winds usually also have a local name. In foreign literature they are called katabatic winds.

** Qualitative considerations on this question were expressed earlier by L. Prandtl ([6]).

(the air density is eliminated by means of the Clapeyron equation). As for $\theta(z)$, for simplicity we set

$$\theta(z) = \theta_{\text{n}} - \gamma(z - H), \tag{3}$$

where H and θ_{n} are the prescribed height of the undisturbed flow and the temperature at its upper boundary. It is known that in a bora the thickness of the cold-air layer is small [(1)]; therefore we shall assume that H does not exceed 3 km.

For the motionless warmer air displaced upward, the equations are

$$RT \frac{dp}{dz} = -pg; \qquad T = T_{\text{n}} - \gamma_1(z - H), \tag{4}$$

analogous to (2), (3). Here, obviously, T, p, and γ_1 are the temperature, pressure, and vertical temperature gradient in the motionless warm air; T_{n} is the prescribed temperature above the interface at $x = -\infty$.

The boundary conditions will be as follows:

$$\begin{aligned} &u = U, \quad \vartheta = p' = 0; \quad \eta = H && \text{for } x = -\infty; \\ &w = u\delta'(x) && \text{for } z = \delta(x); \\ &w = u\eta'(x), \quad P + p' = p && \text{for } z = \eta(x). \end{aligned} \tag{5}$$

In [(5)] it is shown that system (1) reduces to the solution of a single equation containing two arbitrary functions of the stream function ψ ($u = \partial\psi/\partial z$; $w = -\partial\psi/\partial x$). In the problem under consideration, by analogy with [(5)], the form

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of these functions is easily determined from the conditions at $x = -\infty$. As a result we arrive at the equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\mu\lambda}{U^2}\psi = \frac{\mu\lambda}{U} z. \tag{6}$$

The solution of this equation satisfying the conditions $\psi = 0$ for $z = \delta(x)$ and $\psi = UH$ for $z = \eta(x)$ (which follow from (5)) will be

$$\psi = \frac{U^2}{\sqrt{\mu\lambda}} \left\{ z + \frac{1}{\sin h} \left[\zeta \sin(z - \delta) - \delta \sin(\eta - z) \right] \right\}, \tag{7}$$

where $\zeta(x) = H - \eta(x)$; $h(x) = H - \zeta(x) - \delta(x)$. Here, for convenience of calculation, for all quantities denoting vertical dimensions the parameter $U/\sqrt{\mu\lambda}$, which has the dimension of length, has been taken as the scale unit. Thus the quantities H, h, η, ζ, δ, and z appearing in (7) and in all subsequent formulas are dimensionless.

Knowing ψ, one can readily construct formulas for u, w, ϑ, and p'. It remains only to find $\zeta(x)$ (or $h(x)$ and $\eta(x)$). For this purpose we turn to the last of conditions (5), requiring continuity of the pressures at the interface. Substituting the values

$$P(\eta) = P(H) \exp\left(-\frac{gU}{R\sqrt{\mu\lambda}} \int_\eta^H \frac{dz}{\theta} \right); \qquad p(\eta) = P(H) \exp\left(-\frac{gU}{R\sqrt{\mu\lambda}} \int_\eta^H \frac{dz}{T} \right),$$

which follow from (2)–(5), into the last of conditions (5), divided by $P(\eta)$, evaluating the integrals and then expanding the resulting expression in a series in ζ, we obtain

$$\left(\frac{p'}{P} \right)_{z=\eta(x)} \approx -\frac{T_\text{n} - \theta_\text{n}}{T_\text{n}\theta_\text{n}} \frac{gU}{R\sqrt{\mu\lambda}} \zeta. \tag{8}$$

Equating the right-hand sides of (8) and of the Bernoulli equation written for the upper streamline,

$$\left(\frac{p'}{P} \right)_{z=\eta(x)} = \frac{U^2}{2R\theta_\text{n}} \left[1 - \frac{u^2_{z=\eta(x)}}{U^2} - \zeta^2 \right],$$

we obtain, using expression (7), a transcendental equation relating ζ and δ. After elementary transformations, in which the circumstance is taken into account that $\delta = \zeta = 0$ at $x = -\infty$, this equation can be represented in the form:

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Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$\delta = \left(\sqrt{1 + 2\tau\zeta - \zeta^2} - 1\right)\sin h - \zeta\cos h \tag{9}$$

$$\left(\tau = \frac{T_{\text{n}} - \theta_{\text{n}}}{U}\,\frac{\theta_{\text{n}}}{T_{\text{n}}}\sqrt{\frac{\lambda}{\mu}}\right).$$

The parameter τ characterizes the influence of thermal differences between the stationary warm air and the colder intruding air. Equation (9) makes it easy to construct curves $\delta = \delta(\zeta)$ for given τ and H. To do this, we compute δ from (9), assigning various h and ζ, then plot the points (ζ, δ) in the plane of the variables (ζ, δ), assign to these points the values $H = h + \zeta + \delta$, and finally draw isolines of H by interpolation. In this way, for any value of τ of interest, one can construct curves analogous to those in Fig. 1 (we constructed graphs for $\tau = 0.4;\ 0.8;\ \ldots;\ 4.0$, $H = 0.5;\ 1.0;\ \ldots;\ 3.0$). Knowing τ and H, we find graphically, by means of the corresponding curve, the value of ζ for all required values of δ. Thus, the posed problem is solved.

Fig. 1

Fig. 2

We now turn to a brief analysis of the solution obtained. For the curves $\delta = \delta(\zeta)$ constructed by us, it is characteristic that they all emerge from the origin of coordinates ($\delta = \zeta = 0$ at $x = -\infty$), have a maximum, and their other end abuts the ζ-axis.

It follows from this:

1. If the dimensionless height of the mountain is lower than the maximum ordinate of the corresponding curve $\delta = \delta(\zeta)$ (we shall call this ordinate the critical height), then in determining ζ we shall be forced to move along the ascending branch of the curve $\delta = \delta(\zeta)$ from the origin to some point corresponding to the height of the mountain, and back. Therefore, the distribution of the meteorological elements in the vertical above different points of the mountain surface located at the same height will be identical, regardless of whether these points are on one side or on both sides of the summit (if, for example, the mountain is symmetric, then the pattern of streamlines will also be symmetric).

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Fig. 3

Figure 3: Fig. 3

2. If the dimensionless height of the mountain coincides with the critical height, then in calculations for points lying above the windward slope one can use both the ascending and the descending branch of the curve. Consequently, in this case there are two solutions. One of them, in its character, will differ in no way from the solution corresponding to case 1. The other

the solution gives a narrowing of the flow (sometimes by several times) as the air moves down the mountain and a corresponding increase in velocity, i.e., the picture observed in a bora (1)*.

Whether this critical regime is realized in nature or not depends on the distribution of pressure at $x = +\infty$. This makes clear the reason why a bora arises, generally speaking, comparatively rarely (1). For it to arise, certain conditions must be realized in the approaching flow and far from the mountain on the lee side.

Fig. 3

3. If the dimensionless height of the mountain is greater than the critical one, then a solution cannot be constructed. In this case one should expect the formation, on the windward side, of a nonstationary hydraulic jump moving upstream. Downstream of it there will be a critical regime with those initial data (the depth and velocity of the cold layer) that are formed behind this jump.

Let us now give a numerical example. Let $H = 3$ km; $\theta_{\mathrm{n}} = 270°$; $T_{\mathrm{n}} = 280°$; $U = 10$ m/sec; $\lambda = 1/27$ m/sec^2 $\cdot$ deg; $\mu = 3 \cdot 10^{-3}$ deg/m. Then $H \approx 3$ (dimensionless), $\tau \approx 3.2$.

The pattern of streamlines, calculated by formula (7) with the aid of the corresponding curve of Fig. 1 for the case of a mountain whose dimensionless height is equal to the critical height, is shown in Fig. 2. An analogous pattern for a mountain whose height does not reach the critical value is given in Fig. 3. The wind profiles are drawn with a heavy line.

In conclusion we express our deep gratitude to G. K. Sulakvelidze and L. M. Levin, discussions with whom contributed to the formulation of this work.

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* If the pressure near the ground on the lee side lies between the values that correspond to the two indicated solutions, then a hydraulic jump must form there, the same as the one observed in the tailwater of a spillway dam under analogous conditions. According to a communication by A. Gusev, such a phenomenon is indeed observed.

Note: Figure translations are in progress. See original paper for figures.

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