Physics

N. A. CHERNIKOV

1960

SovietRxiv

Abstract

Full Text

Physics

N. A. CHERNIKOV

REDUCTION OF THE RELATIVISTIC COLLISION INTEGRAL TO BOLTZMANN FORM

(Presented by Academician V. A. Fock on 14 III 1960)

In papers ([1,2]) an expression was found for the relativistic collision integral. For the case of elastic collisions we reduce it here to Boltzmann form. This makes it possible to prove that the collision integral for particles distributed according to the relativistic Maxwell law is equal to 0.

Consider collisions of particles of some species α with particles of species β. We shall assume that all collisions of these particles are elastic. Introduce the following notation: $P = (\mathbf{p}, p_4)$ or $P' = (\mathbf{p}', p_4')$ is the four-dimensional momentum of a particle of species α; $\mathbf{p}$ or $\mathbf{p}'$ is the three-dimensional momentum of this particle; $m_\alpha \geq 0$ is the rest mass of a particle of species α; $p_4 = \sqrt{m_\alpha^2 + p^2/c^2}$ or $p_4' = \sqrt{m_\alpha^2 + {p'}^2/c^2}$ is the moving mass of this particle; $Q = (\mathbf{q}, q_4)$ or $Q' = (\mathbf{q}', q_4')$ is the four-dimensional momentum of a particle of species β; $\mathbf{q}$ or $\mathbf{q}'$ is its three-dimensional momentum; $m_\beta \geq 0$ is the rest mass of a particle of species β; $q_4 = \sqrt{m_\beta^2 + q^2/c^2}$ or $q_4' = \sqrt{m_\beta^2 + {q'}^2/c^2}$ is its moving mass; $dP = \dfrac{dp_1\, dp_2\, dp_3}{p_4}$, $dP' = \dfrac{dp_1'\, dp_2'\, dp_3'}{p_4'}$, $dQ = \dfrac{dq_1\, dq_2\, dq_3}{q_4}$, $dQ' = \dfrac{dq_1'\, dq_2'\, dq_3'}{q_4'}$; for any four-dimensional vectors $X = (\mathbf{x}, x_4)$ and $Y = (\mathbf{y}, y_4)$,

$$(X, Y) = x_4 y_4 - \frac{\mathbf{xy}}{c^2}, \qquad \langle X, Y \rangle = c\sqrt{(X,Y)^2 - (X,X)(Y,Y)};$$

$$\Delta\sigma = h_{\alpha\beta}(\langle P, Q\rangle, \cos\theta^*) \sin\theta^*\, d\theta^*\, d\varphi^* \tag{1}$$

is the differential cross section for scattering of a particle of species α with momentum P by a particle of species β with momentum Q in the system of their center of inertia*.

In an arbitrary reference system the differential scattering cross section can be written in the form

$$d\sigma = H_{\alpha\beta}(\langle P, Q\rangle, \langle P', Q\rangle, \langle P', P\rangle)\, dP',$$

where

$$H_{\alpha\beta}(\langle P,Q\rangle,\langle P',Q\rangle,\langle P',P\rangle)=\frac{(P+Q,P+Q)}{\langle P,Q\rangle c^2}\,h_{\alpha\beta}\left(\langle P,Q\rangle,\,1+\frac{(P+Q,P+Q)(P,P-P')c^2}{\langle P,Q\rangle^2}\right)$$
$$\times\,\delta((P-P',Q+P)). \tag{2}$$

As shown in paper ([2]), the relativistic collision integral for particles of species α with particles of species β has the form

$$I_{\alpha\beta}=I_{\alpha\beta}^{(1)}-I_{\alpha\beta}^{(2)}, \tag{3}$$

$$I_{\alpha\beta}^{(1)}=\int_{-\infty}^{\infty}\cdots\int A_\alpha(\mathbf{p}')\,A_\beta(\mathbf{q}')\,\langle P',Q'\rangle\,H_{\alpha\beta}(\langle P',Q'\rangle,\langle P,Q'\rangle,\langle P,P'\rangle)\,dP'\,dQ',$$

$$I_{\alpha\beta}^{(2)}=\int_{-\infty}^{\infty}\cdots\int A_\alpha(\mathbf{p})\,A_\beta(\mathbf{q})\,\langle P,Q\rangle\,H_{\alpha\beta}(\langle P,Q\rangle,\langle P',Q\rangle,\langle P',P\rangle)\,dP'\,dQ.$$

* The quantity $\langle P,Q\rangle$ is equal to the product of the rest mass of the colliding particles as a whole and the momentum of one of these particles in the system of their center of inertia.

In the integral $I_{\alpha\beta}^{(1)}$ we introduce the momentum Q $((Q,Q)=m_\beta^2,\ q_4>0)$ in such a way that the equality

$$P+Q=\lambda(P'+Q') \tag{4}$$

is satisfied. It follows that

$$\lambda=\frac{(P,\ P'+Q')+\sqrt{(P,\ P'+Q')^2+(m_\beta^2-m_\alpha^2)(P'+Q',\ P'+Q')}}{(P'+Q',\ P'+Q')}. \tag{5}$$

We shall identify the obtained momentum Q with the momentum Q appearing in the integral $I_{\alpha\beta}^{(2)}$. In addition, the momentum P' from the first integral will be identified with the momentum P' from the second integral. It is not difficult to show that, for the function $H_{\alpha\beta}$ defined by formula (2), the equality

$$\langle P',Q'\rangle H_{\alpha\beta}^{[[unclear:superscriptoverH]]}(\langle P',Q'\rangle,\ \langle P,Q'\rangle,\ \langle P,P'\rangle)=$$

$$=\langle P,Q\rangle H_{\alpha\beta}(\langle P,Q\rangle,\ \langle P',Q\rangle,\ \langle P',P\rangle). \tag{6}$$

Equality (4) is satisfied if, from the integration variables $\mathbf{p}',\mathbf{q}'$ in the integral $I_{\alpha\beta}^{(1)}$, we pass to the variables $\vec{\chi}$ and $\mathbf{q}$:

$$\mathbf{p}' = \vec{\chi} + \mathbf{u}\chi_\alpha + \frac{\mathbf{u}(\mathbf{u}\vec{\chi})}{(u_4+1)c^2}, \tag{7a}$$

$$\mathbf{q}' = -\vec{\chi} + \mathbf{u}\chi_\beta - \frac{\mathbf{u}(\mathbf{u}\vec{\chi})}{(u_4+1)c^2}, \tag{7b}$$

where

$$\mathbf{u} = \frac{\mathbf{p}+\mathbf{q}}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P,Q)}},$$

$$u_4 = \sqrt{1+\frac{u^2}{c^2}} = \frac{p_4+q_4}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P,Q)}}, \tag{8}$$

$$\chi_\alpha = \sqrt{m_\alpha^2 + \frac{\chi^2}{c^2}}, \qquad \chi_\beta = \sqrt{m_\beta^2 + \frac{\chi^2}{c^2}},$$

since from (7) it follows that

$$p_4' = u_4\chi_\alpha + \frac{\mathbf{u}\vec{\chi}}{c^2}, \qquad q_4' = u_4\chi_\beta - \frac{\mathbf{u}\vec{\chi}}{c^2}, \tag{9}$$

and hence

$$P' + Q' = \frac{\chi_\alpha + \chi_\beta}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P,Q)}}(P+Q). \tag{10}$$

Formula (7a) expresses the momentum $\mathbf{p}'$ of a particle of species α in the rest frame through its momentum $\vec{\chi}$ in a frame moving with velocity $\mathbf{u}$. Formula (7b) expresses the momentum $\mathbf{q}'$ of a particle of species β in the rest frame through its momentum $-\vec{\chi}$ in a frame moving with velocity $\mathbf{u}$ (see, for example, ([3])).

The Jacobian of the transformation (7) can be represented in the form of the following product:

$$\frac{D(p_1',p_2',p_3',q_1',q_2',q_3')}{D(\chi_1,\chi_2,\chi_3,q_1,q_2,q_3)} = \frac{D(p_1',p_2',p_3',q_1',q_2',q_3')}{D(\chi_1,\chi_2,\chi_3,u_1,u_2,u_3)}\,\frac{D(u_1,u_2,u_3)}{D(q_1,q_2,q_3)}. \tag{11}$$

The second factor in this product is not difficult to find. It is equal to

$$\frac{D(u_1,u_2,u_3)}{D(q_1,q_2,q_3)} = \frac{u_4}{q_4}\,\frac{m_\beta^2 + (P,Q)}{\left[m_\alpha^2 + m_\beta^2 + 2(P,Q)\right]^2}. \tag{12}$$

In the Jacobian

$$J = \frac{D(p_1', p_2', p_3', q_1', q_2', q_3')}{D(x_1, x_2, x_3, u_1, u_2, u_3)}$$

to the fourth row, containing the derivatives with respect to q_1', we add the first row. Similarly, to the fifth row we add the second, and to the sixth—the third. As a result this Jacobian is reduced to the form

$$J = (x_\alpha + x_\beta)^3 \begin{vmatrix} A & B \\ C & E \end{vmatrix}, \tag{13}$$

where

$$A = \left(\frac{\partial p_i'}{\partial x_k}\right), \qquad B = \left(\frac{\partial p_i'}{\partial u_k}\right), \qquad C = \left(\frac{u_i x_k}{x_\alpha^2 x_\beta^2 c^2}\right)$$

are matrices of the third order; E is the identity matrix. Adding suitable linear combinations of the last three columns to the first three columns of determinant (13), we find

$$\frac{J}{(x_\alpha + x_\beta)^3} = \begin{vmatrix} A - BC & B \\ 0 & E \end{vmatrix} = |A - BC| =$$

$$= \left| \frac{\partial p_i'}{\partial x_k} - \frac{x_k}{x_\alpha x_\beta c^2} \sum_{j=1}^{3} u_j \frac{\partial p_i'}{\partial u_j} \right| = \frac{p_4' q_4'}{u_4 x_\alpha x_\beta}. \tag{14}$$

With the aid of equalities (11), (12), and (14) it is not difficult to establish that, for $(P - P',\, Q + P) = 0$, the Jacobian of transformation (7) from the variables $\mathbf{p}, \mathbf{q}'$ to the variables $\vec{x}, \mathbf{q}$ is equal to

$$\frac{p_4' q_4'}{x_\alpha q_4}.$$

Combining this result with equality (6), we write the integral $I_{\alpha\beta}^{(1)}$ in the form

$$I_{\alpha\beta}^{(1)} = \int_{-\infty}^{\infty} \cdots \int A_\alpha(\mathbf{p}')A_\beta(\mathbf{q}')\langle P, Q\rangle H_{\alpha\beta}(\langle P, Q\rangle, \langle P', Q\rangle, \langle P', P\rangle)\frac{dx_1 dx_2 dx_3}{x_\alpha}\, dQ. \tag{15}$$

In the integral $I_{\alpha\beta}^{(2)}$, from the variables of integration p'_1, p'_2, p'_3 we pass to the variables x_1, x_2, x_3 by formulas (7a) and (8):

$$I_{\alpha\beta}^{(2)} = \int_{-\infty}^{\infty} \cdots \int A_\alpha(\mathbf{p})A_\beta(\mathbf{q})\langle P, Q\rangle H_{\alpha\beta}(\langle P, Q\rangle, \langle P', Q\rangle, \langle P', P\rangle)\times$$

$$\times \frac{dx_1 dx_2 dx_3}{x_\alpha}\, dQ. \tag{16}$$

We note that

$$\delta((P - P',\, Q + P)) =$$

$$= \frac{1}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P, Q)}}\, \delta\left(x_\alpha - \frac{m_\alpha^2 + (P, Q)}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P, Q)}}\right). \tag{17}$$

In the integrals (15) and (16), from the variables x_1, x_2, x_3 we pass to the spherical variables x, θ, φ and, using equality (17), carry out the integration with respect to x. After this we can write the collision integral (3) in Boltzmann form:

$$I_{\alpha\beta} = \int \cdots \int (A'_\alpha A'_\beta - A_\alpha A_\beta)\langle P, Q\rangle h_{\alpha\beta}\left(\langle P, Q\rangle, \frac{\vec{x}\vec{x}_0}{x x_0}\right) \sin\theta\, d\theta\, d\varphi\, dQ, \tag{18}$$

where

$$A_\alpha = A_\alpha(\mathbf{p}), \qquad A_\beta = A_\beta(\mathbf{q}),$$

$$A'_\alpha = A_\alpha\left(\vec{x} + \mathbf{u}x_\alpha + \frac{\mathbf{u}(\mathbf{u}\vec{x})}{(u_4 + 1)c^2}\right), \qquad A'_\beta = A_\beta\left(-\vec{x} + \mathbf{u}x_\beta - \frac{\mathbf{u}(\mathbf{u}\vec{x})}{(u_4 + 1)c^2}\right), \tag{19}$$

$$\vec{x}_0 = \mathbf{p} - \mathbf{u}p_4 + \frac{\mathbf{u}(\mathbf{u}\mathbf{p})}{(u_4 + 1)c^2}, \qquad x = x_0 = \frac{\langle P, Q\rangle}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P, Q)}},$$

$$x_\alpha = \frac{m_\alpha^2 + (P,Q)}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P,Q)}}, \qquad x_\beta = \frac{m_\beta^2 + (P,Q)}{\sqrt{m_\alpha^2 + m_\beta^2 + 2(P,Q)}}. \tag{20}$$

$\mathbf{u}$ and u_4 are determined by formulas (8). The region of integration in (18) is specified by the inequalities $-\infty < q_1, q_2, q_3 < \infty,\ 0 \leqslant \varphi < 2\pi,\ 0 \leqslant \theta \leqslant \pi$.

The integral (18) vanishes if

$$A_\alpha(\mathbf{p}) = c_\alpha \exp\{-(R,P)\}, \qquad A_\beta(\mathbf{q}) = c_\beta \exp\{-(R,Q)\}, \tag{21}$$

where c_α, c_β are scalars and R is a four-dimensional vector independent of $\mathbf{p}$ and $\mathbf{q}$, since in this case $A'_\alpha A'_\beta - A_\alpha A_\beta = 0$. From the condition of convergence of the integral

$$\iiint\limits_{-\infty}^{\infty} A_\alpha(\mathbf{p})\, dp_1 dp_2 dp_3$$

it follows that R is a time-like vector.

Changing the notation, we write (21) in the form

$$A_\alpha(\mathbf{p}) = a_\alpha \exp\{-b[(\Omega,P) - m_\alpha]c^2\},$$

$$A_\beta(\mathbf{q}) = a_\beta \exp\{-b[(\Omega,Q) - m_\beta]c^2\}, \tag{22}$$

where $b > 0,\ \omega_4 > 0,\ (\Omega,\Omega) = 1$. The vector $\Omega = (\vec{\omega}, \omega_4)$ is the mean four-dimensional velocity of the gas at the point under consideration. Indeed, let us consider the ratio of integrals

$$\bar{\mathbf{v}} = \iiint\limits_{-\infty}^{\infty} \frac{\mathbf{p}}{p_4} \exp\{-b(\Omega,P)c^2\}\, dp_1\, dp_2\, dp_3 \times \left[\iiint\limits_{-\infty}^{\infty} \exp\{-b(\Omega,P)c^2\}\, dp_1\, dp_2\, dp_3\right]^{-1}. \tag{23}$$

We pass from the variables of integration $\mathbf{p}$ to $\mathbf{s}$ by formula (7a), with the change of notation $\mathbf{p}' \to \mathbf{p},\ \mathbf{u} \to \vec{\omega},\ \vec{x} \to \mathbf{s}$. To transform the denominator we shall also use formula (9). As a result we obtain

$$\bar{\mathbf{v}} = \iiint\limits_{-\infty}^{\infty} \mathbf{p} \exp\{-bc^2 s_4\} \frac{ds_1\, ds_2\, ds_3}{s_4} \times \left[\iiint\limits_{-\infty}^{\infty} p_4 \exp\{-bc^2 s_4\} \frac{ds_1\, ds_2\, ds_3}{s_4}\right]^{-1} = \frac{\vec{\omega}}{\omega_4}, \tag{24}$$

which was required to be proved.

The distribution (22) is a generalization of the law of the local Maxwell distribution to the relativistic case. The quantity $[(\Omega, P) - m]c^2$ is the kinetic energy of a particle in a frame of reference moving with velocity Ω.

Joint Institute
for Nuclear Research

Received
27 II 1960

CITED LITERATURE

1. N. A. Chernikov, *DAN*, **114**, No. 3, 530 (1957).

2. N. A. Chernikov, *Scientific Reports of Higher Schools, Physics-Mathematics Series*, No. 1, 168 (1959).

3. V. A. Fock, *The Theory of Space, Time and Gravitation*, 1955.

Note: Figure translations are in progress. See original paper for figures.