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**Abstract**

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## MATHEMATICS

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# AN EXACT ESTIMATE OF THE ERROR OF MULTIDIMENSIONAL QUADRATURE FORMULAS FOR FUNCTIONS OF THE CLASS $S_p$

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**Introduction.** The error of an arbitrary quadrature formula

$$\delta(f) = \int_K f(P)\,dP - \sum_{i=1}^{N} C_i f(P_i)$$

is a linear functional, which we shall regard as given on some linear normed space $H$ of functions defined on the unit $d$-dimensional cube $K$. The norm $\|\delta\|$ of this functional depends on the nodes $P_1, \dots, P_N$ and the weights $C_1, \dots, C_N$. Generally speaking, any problem of choosing the best quadrature formula (under any conditions) for the whole class $H$ reduces to the problem of the minimum of $\|\delta\|$ (under the corresponding conditions). A survey of results on extremal problems in the theory of quadrature formulas is given in the book of S. M. Nikol' skii ([2]). All these results concern only the one-dimensional case ($d = 1$).

In the present work formulas with equal weights $C_1 = \dots = C_N = 1/N$ are studied. It has proved possible to compute the norm of the error (for any $d$) for classes $S_p$ of functions of many variables ([4]). The geometric meaning of the dependence of the norm on the nodes is clarified. The question of the best order of convergence of quadrature formulas for the classes $S_p$ and for the classes $H_\alpha$ embedded in them is considered. For the class $S_1$ a two-dimensional extremal problem is solved.

**1. Main theorem.** Choose arbitrary points $P_1, \dots, P_N$ from $K$. Denote the coordinates of $P_i$ by $x_{i1}, \dots, x_{id}$. Introduce the function $\varphi_q(P_1, \dots, P_N)$, which will play the main role below.

**Definition***.

$$\varphi_q(P_1, \dots, P_N) = \sup_m{}' \left\{ \sum_j \left| \sum_{i=1}^{N} \operatorname{sgn}\left[\chi_{k_1}(x_{i1}) \cdots \chi_{k_d}(x_{id})\right] \right|^q \right\}^{1/q}. \qquad (1)$$

The prime indicates that the case $m_1 = \dots = m_d = 0$ is excluded when the upper bound is taken.

Let us estimate the error**

$$\delta_N(f) = \int_K f(P)\,dP - \frac{1}{N}\sum_{i=1}^{N} f(P_i). \tag{2}$$

* All notation connected with the Haar functions $\chi_k(x)$ corresponds to article ([4]). For brevity, instead of $(m_1, \dots, m_d)$, $(j_1, \dots, j_d)$, $(k_1, \dots, k_d)$ we shall write $m, j, k$.

** It would be more correct to write $\delta(f; P_1, \dots, P_N)$.

**Theorem.** Whatever the points $P_1, \dots, P_N$ in $K$, for the class of functions $S_p$

$$\|\delta_N\| = \frac{\varphi_q(P_1, \dots, P_N)}{N}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3}$$

**Outline of the proof.** First, it is necessary to establish that

$$|\delta_N(f)| \le \|f\|_p \varphi_q(P_1, \dots, P_N)/N. \tag{4}$$

Expand $f(P)$ in the uniformly convergent Haar series

$$f(P) = \sum_m \sum_i c_k \chi_{k_1}(x_1) \cdots \chi_{k_d}(x_d)$$

and substitute this series into (2). Interchanging the order of summation with respect to $i$ and with respect to $(m, j)$, we estimate $|\delta_N(f)|$, applying Hölder' s inequality to the sum over $j$. We obtain the inequality

$$|\delta_N(f)| \le \frac{1}{N} \sum_m{}' \prod_{s=1}^{d} 2^{\frac{\widetilde{m}_s - 1}{2}} \left\{ \sum_j |c_k|^p \right\}^{\frac{1}{p}} \left\{ \sum_j \left| \sum_{i=1}^{N} \operatorname{sgn}[\chi_{k_1}(x_{i1}) \cdots \chi_{k_d}(x_{id})] \right|^q \right\}^{\frac{1}{q}},$$

from which (4) follows.

Second, it is necessary to construct a function $g(P) \in S_p$ for which inequality (4) becomes an equality. Fix a value $\widetilde{m} = (\widetilde{m}_1, \dots, \widetilde{m}_d)$ at which the least upper bound in formula (1) is attained (the existence of such an $\widetilde{m}$ follows from item 2c). Denote

$$A_{\widetilde{m}j} = \sum_{i=1}^{N} \operatorname{sgn}[\chi_{\widetilde{m}_1 j_1}(x_{i1}) \cdots \chi_{\widetilde{m}_d j_d}(x_{id})].$$

It is easy to verify that the function possessing the required property is

$$g(P) = \sum_{j} \operatorname{sgn} A_{\widetilde{m}j}\, |A_{\widetilde{m}j}|^{q-1} \chi_{\widetilde{m}_1 j_1}(x_1) \cdots \chi_{\widetilde{m}_d j_d}(x_d).$$

**2. Geometric definition of $\varphi_q(P_1, \dots, P_N)$.** We shall regard the net $P_1, \dots, P_N$ as fixed. Whatever the volume $V \subset K$, by $S_N(V)$ we shall denote the number of points of the net lying in $V$.

a) Suppose that all $m_s > 0$. By fixing the system of indices $m = (m_1, \dots, m_d)$, we thereby fix a partition of $K$ into parallelepipeds $\Pi_k \equiv \Pi_{k_1 \dots k_d} = l_{k_1} \times \cdots \times l_{k_d}$, where $l_{k_s} = l_{m_s j_s}$. The sum over $j = (j_1, \dots, j_d)$ is the sum over all parallelepipeds of the given partition.

Move the origin of coordinates to the center $P'$ of the parallelepiped $\Pi_k$. Let the new coordinates be $\xi_s$. The coordinate planes $\xi_s = 0$ divide $\Pi_k$ into $2^d$ parallelepipeds. The sum (in the set-theoretic sense) of those parallelepipeds in which $\operatorname{sgn}(\xi_1 \cdots \xi_d) = (-1)^d$ will be called the positive part of $\Pi_k$ and denoted by $V_k^+$; the remaining parallelepipeds constitute the negative part $V_k^-$. Obviously, the volumes of these parts are equal.

If $P_i \in V_k^+$, then

$$\operatorname{sgn}[\chi_{k_1}(x_{i1}) \cdots \chi_{k_d}(x_{id})] = (-1)^d \operatorname{sgn}(\xi_{i1} \cdots \xi_{id}) = 1.$$

If $P_i \in V_k^-$, then this expression is equal to $-1$, while if $P_i \notin \Pi_k$, it vanishes. Consequently,

$$\sum_{i=1}^{N} \operatorname{sgn}[\chi_{k_1}(x_{i1}) \cdots \chi_{k_d}(x_{id})] = S_N(V_k^+) - S_N(V_k^-).$$

b) We shall call the quantity

$$\left\{ \sum_{j_1, \dots, j_d} \left| \sum_{i=1}^{N} \operatorname{sgn}\left[\chi_{k_1}(x_{i1}) \cdots \chi_{k_d}(x_{id})\right] \right|^q \right\}^{1/q} \tag{5}$$

in the case when all $m_s > 0$, the $d$-dimensional discrepancy of the points $P_1, \dots, P_N$ with respect to the partition $(m_1, \dots, m_d)$ of the $d$-dimensional cube $K$. Consider the partition $(0, m_2, \dots, m_d)$, where all $m_s$, except the first, are greater than zero. It is easy to see that expression (5) in this case is equal to

$$\left\{ \sum_{j_2,\dots,j_d} \left| \sum_{i=1}^{N} \operatorname{sgn}\left[\chi_{k_2}(x_{i2}) \cdots \chi_{k_d}(x_{id})\right] \right|^q \right\}^{1/q}$$

and is the $(d-1)$-dimensional discrepancy of the projections of the points $P_1, \dots, P_N$ onto the face $x_1 = 0$ of the cube $K$ with respect to the partition $(m_2, \dots, m_d)$ of this $(d-1)$-dimensional face.

In order to compute $\varphi_q(P_1, \dots, P_N)$, it is necessary to find the greatest $d$-dimensional discrepancy of the points $P_1, \dots, P_N$ in $K$, as well as the greatest $s$-dimensional discrepancies of the projections of the points $P_1, \dots, P_N$ onto all $s$-dimensional coordinate "faces" of the cube $K$, $1 \le s \le d-1$. The function $\varphi_q(P_1, \dots, P_N)$ is equal to the largest of all these discrepancies.

c) It is not difficult to verify that $\varphi_q(P_1, \dots, P_N)$ coincides with the function $\varphi_q(N)$, geometrically constructed in (1). The theorem of item 1 generalizes and refines Theorem 4.1 from ([1]).

From the geometric definition some properties of $\varphi_q$ are readily derived. We note that the upper bound in formula (1) is in fact taken only over a finite number of partitions. The function $\varphi_q$ is bounded: for any $P_1, \dots, P_N$ from $K$,

$$N^{1/q} \le \varphi_q(P_1, \dots, P_N) \le N. \tag{6}$$

3. **On the best grids for $S_p$.** In order to choose the best grid for a given number of nodes $N$, it is necessary to find $\inf \varphi_q(P_1, \dots, P_N)$ over all possible grids $P_1, \dots, P_N$. This has so far been accomplished only in two special cases: a) for $d = 1$, $\inf \varphi_q(P_1, \dots, P_N) = N^{1/q}$ (cf. (6)); an evenly spaced grid may be taken as extremal; b) for $d = 2$, $\inf \varphi_\infty(P_1, \dots, P_N) = 2$; consequently, the two-dimensional grids constructed in ([1]), for which $\varphi_q = 2^{1/p} N^{1/q}$, are extremal with respect to the class $S_1$.

Such two-dimensional grids can be constructed by the formulas

$$x_{ij} = y_{ji} = \frac{1}{n}(i + p_j), \qquad 0 \le i, j \le n-1.$$

Here the number of nodes is $N = 4^k$; $n = \sqrt{N}$; $\{p_j\}$ is an infinite sequence of numbers, $p_0 = 0$; $p_n = 1/2n$ for $n = 2^k$; $p_{n+i} = p_n + p_i$ for $1 \le i \le n-1$.*

4. **On the best order of convergence for $S_p$.** One may pose a simpler problem: to find a family of grids (containing grids with arbitrarily large numbers of nodes) that ensures the best order of decrease of the error as $N \to \infty$. According to (6), the best order of growth of $\varphi_q$ cannot be less than $N^{1/q}$. Grids for which $\varphi_q \asymp N^{1/q}$ have been constructed only for $d = 1$ and $d = 2$. The very fact of their existence for $d \ge 3$ has not been proved. However, with the aid of the results of ([3]), one can indicate

multidimensional grids (for any $d$) for which $\varphi_q \leq CN^{(1/q)+\varepsilon}$, with $\varepsilon > 0$ arbitrarily small.

We note that, for the study of the order of convergence, it is sufficient to study $\varphi_\infty$—the simplest among all $\varphi_q$. Indeed, from (1) it is not difficult

* *Note added in proof.* An analogous grid was constructed in Roth' s paper ([5]) without connection with the problem of computing integrals.

derive that

$$\varphi_q(P_1, \dots, P_N) \leqslant N^{1/q} \varphi_\infty(P_1, \dots, P_N). \tag{7}$$

Therefore, from $\varphi_\infty \leqslant CN^\varepsilon$ it follows that $\varphi_q \leqslant CN^{(1/q)+\varepsilon}$ for all $q$.

**5. Estimate of the error for functions of the class $H_\alpha$.** In (4) the classes of functions $H_\alpha$ ($0 < \alpha \leqslant 1$) are defined; they are analogues of the one-dimensional classes $\operatorname{Lip} \alpha$, and the embedding theorem is proved: if $\alpha p > 1$, then $H_\alpha \subset S_p$.

**Theorem.** Whatever the points $P_1, \dots, P_N$ from $K$ and the function $f(P)$ from $H_\alpha$ may be,

$$|\delta_N(f)| \leqslant B \, \frac{\varphi_\infty(P_1, \dots, P_N) \ln^d N}{N^\alpha}, \tag{8}$$

where $B \to L(e/2\alpha + 1d \ln 2)^d$ as $N \to \infty$.

**Sketch of proof.** From inequalities (4), (7), and the estimate for $\|f\|_p$ given in the embedding theorem, we obtain an estimate for $|\delta_N(f)|$ containing $p$ as a parameter, with $0 < 1/p < \alpha$. For large $N$ this estimate has its minimum at $1/p = \alpha - d \ln^{-1} N + O(\ln^{-2} N)$, which is equal to the right-hand side of (8).

Estimate (8) is not sharp. However, there is an example showing that $\ln^d N$ cannot be replaced by $\ln^{d-1} N$ in (8).

From this theorem and item 4 it follows that there exist quadrature formulas ensuring, for the class $H_\alpha$, the order of convergence $1/N^{\alpha-\varepsilon}$. It is not difficult to prove that the order of convergence for the class $H_\alpha$ cannot be better than $1/N^\alpha$ (even if formulas with weights are used).

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*Note: Figure translations are in progress. See original paper for figures.*