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A. L. GARKAVI

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Abstract

Full Text

MATHEMATICS

A. L. GARKAVI

ON SPACES OF LINEAR OPERATORS

(Presented by Academician A. N. Kolmogorov on 29 I 1960)

The main result of the work consists in proving that the space of linear operators mapping one Banach space into another is a conjugate space,* provided that the space of images is a conjugate space (Theorem 2).

Let E be a conjugate Banach space. By *E we denote a Banach space (generally speaking, not unique) to which E is conjugate. In the rest we adhere to generally accepted notation.

The following theorem, which will play an auxiliary role below, is also of independent interest.

Theorem 1. Let E_α ($\alpha \in A$) be a certain family of Banach spaces and let E be their direct product, in which the norm

$$\|F\| = \sup_{\alpha \in A} \|F_\alpha\|_{E_\alpha}$$

is introduced, where F_α is the α -th coordinate of the element $F \in E$. If each E_α is a conjugate space, then the space E is conjugate to the space *E , whose elements are all series of the form

$$\varphi = \sum_{k=1}^{\infty} f_{\alpha_k}, \quad \text{for which } \sum_{k=1}^{\infty} \|f_{\alpha_k}\|_{{}^*E_{\alpha_k}} < \infty, \quad (1)$$

where $\alpha_k \in A$, $f_{\alpha_k} \in {}^*E_{\alpha_k}$.

Moreover,

$$\|\varphi\|_{{}^*E_1} = \sum_{k=1}^{\infty} \|f_{\alpha_k}\|_{{}^*E_{\alpha_k}}.$$

According to Alaoglu' s criterion ⁽¹⁾, in order to prove the conjugacy of E it is sufficient to show that in the space E^* , conjugate to E , there exists a set Γ satisfying the conditions:

- 1) Γ is total, i.e., if $\varphi(F) = 0$ for every $\varphi \in \Gamma$, then $F = \theta$.
- 2) The linear span Γ_L of the set Γ is dual, i.e., for every $F \in E$ the relation

$$\|F\| = \sup_{\varphi \in \Gamma_L} \frac{|\varphi(F)|}{\|\varphi\|_{E^*}} \quad (\varphi \neq \theta)$$

holds.

- 3) The closed unit sphere S_E of the space E is bicomact in the topology $\sigma(E, \Gamma)$, i.e., the topology in which neighborhoods of the point F_0 are defined by the inequalities:

$$|\varphi_k(F - F_0)| < \varepsilon, \quad \varphi_k \in \Gamma, \quad k = 1, 2, \dots, n \quad (n < \infty).$$

* That is, a space conjugate to some Banach space.

Under these conditions the space E turns out to be conjugate to the closure $\bar{\Gamma}_L$ of the set Γ_L in the metric E^* .

We show that in our case one may take as the set Γ the set of functionals of the form $\varphi_\alpha, f_\alpha(F) = f_\alpha(F_\alpha)$, where $\alpha \in A$, and f_α ranges over the set of all elements of the space ${}^*E_\alpha \subset E_\alpha^*$. In verifying condition 3) we rely on Tikhonov's theorem on the bicomactness of a topological product of bicomact spaces.

Corollary. The space of real functions $\psi(t)$, bounded on the interval $[0, 1]$, with norm

$$\|\psi\| = \sup_{0 \leq t \leq 1} |\psi(t)|$$

is conjugate to the subspace of the space of functions of bounded variation consisting of step functions that vanish at the point 0 and are continuous from the right at every interior point of the interval $[0, 1]$.*

Theorem 2. Let $B = B(X \rightarrow Y)$ be the space of linear operators mapping the Banach space X into the Banach space Y . Then, if Y is a conjugate space, the space B is also conjugate.

Let E be the space of bounded functions $F(x)$, defined on the unit sphere S_X , with values in the space Y and with norm

$$\|F\| = \sup_{x \in S_X} \|F(x)\|_Y.$$

It is clear that the space B is equivalent to the subspace B_0 of the space E consisting of restrictions T_0 of linear operators $T \in B$ to the sphere S_X . Since E belongs to the type of spaces considered in Theorem 1 (with $A = S_X, \alpha = x \in S_X, F_x = F(x), E_x = Y$), E is a conjugate space. To prove the conjugacy of the subspace B_0 it is enough, according to the results of M. G. Krein and V. V. Shmulian⁽³⁾, to show that the unit sphere S_{B_0} is regularly convex, or, which is the same in this case, bicomact in the topology $\sigma(E, {}^*E)$. Here, as indicated in Theorem 1, we may regard ${}^*E \subset E^*$ as the closed linear envelope of the set $\Gamma \subset E^*$, consisting of functionals $\varphi_{x,f}$ having the form $\varphi_{x,f}(F) = f[F(x)]$ ($f \in {}^*Y, x \in S_X$). Using the totality of the set *Y (relative to Y), one can show that S_{B_0} is closed in the topology $\sigma(E, \Gamma)$. But, as follows from Alaoglu's results⁽¹⁾, the topology $\sigma(E, \Gamma)$ coincides with the topology $\sigma(E, {}^*E)$, and since

S_{B_0} belongs to the sphere S_E , which is bicomact in the topology $\sigma(E, {}^*E)$, it follows that S_{B_0} , by virtue of its closedness in this topology, is also a bicomact set. Consequently, the space B_0 , or, what is the same, the space B , is conjugate.

Remark. Using the results of (2) and taking into account representation (1), we conclude that the space *B is equivalent to the quotient space ${}^*E/\perp B_0$, where $\perp B_0$ is the annihilator of the subspace B_0 in the space *E , and every regular functional φ on the space B ($\varphi \in {}^*B \subset B^*$) is representable (not uniquely) in the form

$$\varphi(T) = \sum_{k=1}^{\infty} f_k(Tx_k) \left(\sum_{k=1}^{\infty} \|f_k\|_{Y} < \infty \right), \quad (2)$$

where $f_k \in {}^*Y$, $x_k \in S_X$, $T \in B$.

The conjugacy of spaces of operators of the type under consideration ensures the existence of solutions of a number of extremal problems in these spaces. We indicate, in particular, the following two theorems.

Theorem 3. For every bounded function F , defined on the unit sphere S_X of the space X with values in the conjugate pro-

* This proposition in essence also follows from the results of (2).

space Y there exists a nearest linear operator $T^* \in B = B(X \rightarrow Y)$, i.e. an operator T^* such that

$$\sup_{x \in S_X} \|F(x) - T^*(x)\|_Y = \inf_{T \in B} \sup_{x \in S_X} \|F(x) - T(x)\|_Y.$$

Theorem 4. Let $\{x_n\}$ be an arbitrary sequence of elements of the space X , let $\{y_n\}$ be a bounded sequence of elements of the conjugate space Y , and let S_B^K be the set of linear operators $T \in B = B(X \rightarrow Y)$ whose norms are bounded by the number K . There exists a linear operator $T^* \in S_B^K$ realizing the best approximate solution, in the sense of Chebyshev, of the (inconsistent) system of equations

$$Tx_n = y_n \quad (n = 1, 2, \dots) \quad \text{under the condition } \|T\|_B \leq K,$$

i.e. an operator T^* such that

$$\sup_n \|T^*x_n - y_n\|_Y = \inf_{T \in S_B^K} \sup_n \|Tx_n - y_n\|_Y.$$

Let us also note that the author's results on the existence of the best N -net and the best N -section of a bounded set in the conjugate space carry over to spaces of operators of the type under consideration (4).

Recall (see, for example, ⁽⁵⁾) that the weak topology of the space $B = B(X \rightarrow Y)$, as a space of operators, is defined as the topology in which a neighborhood of a point T_0 is given by inequalities of the form

$$|f_k(Tx_k - T_0x_k)| < \varepsilon \quad (k = 1, 2, \dots, n; n < \infty),$$

where $x_k \in X$, $f_k \in Y^*$.

From what has been said above it follows, in this way, that in the case where the space Y is reflexive, the topology $\sigma(B, {}^*B)$ coincides with the weak topology of the space B as a space of linear operators. Therefore the following holds.

Theorem 5. If the space Y is reflexive, then the space of linear operators $B(X \rightarrow Y)$ is weakly complete, and its unit sphere is weakly bicomact; see also ⁽⁷⁾.

Theorem 6. If the space X is separable, and the space Y is reflexive or conjugate to a separable space, then the unit sphere S_B of the space of linear operators $B(X \rightarrow Y)$ is sequentially compact in the topology $\sigma(B, {}^*B)$ (and in the case where Y is reflexive, also in the weak topology of B as a space of operators).

To conjugate spaces of linear operators there applies, of course, the Krein-Milman theorem on the coincidence of a regularly convex bounded set with the regularly convex hull of its extreme points ⁽⁶⁾. Using this theorem and taking into account the representation (2), we obtain the following proposition:

Theorem 7. 1) If the space Y is conjugate, then for any operator T belonging to the unit sphere S_B of the space $B(X \rightarrow Y)$, and arbitrary $x \in X$ and $f \in {}^*Y$, the relation

$$f(Tx) = \int_R f(\tilde{T}x) d\mu_T(e), \quad (3)$$

holds, where R is the set of extreme operators T of the unit sphere S_B ; $\mu_T(e)$ is a certain nonnegative additive function of a set $e \subset R$ such that $\mu_T(R) = 1$.

2) If the space Y is reflexive, then, for each fixed $x \in X$, the set of all points $\{Tx\}$, $T \in S_B$, coincides with the closed convex hull of the points $\{\tilde{T}x\}$, where $\tilde{T} \in R$, and for every $T \in S_B$ and $x \in X$ the relation

$$Tx = \int_R \tilde{T}x d\mu_T(e),$$

holds, where the integral is understood in the Pettis-Stieltjes sense, i.e. in the sense that equality (3) is satisfied for every $f \in Y^*$.

As follows from Theorem 2, the class of conjugate spaces of linear operators turns out to be very broad. In this connection it is interesting to note that even the space of linear operators acting in a separable Hilbert space does not possess the property of reflexivity.

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Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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References Cited

1. L. Alaoglu, *Ann. of Math.*, **41**, 252 (1940).
2. A. P. Artemenko, *Matem. sborn.*, **6**, 215 (1939).
3. M. G. Krein, V. A. Smulian, *Ann. of Math.*, **41**, 556 (1940).
4. A. L. Garkavi, *UMN*, **15**, 2 (1960).
5. E. Hille, *Functional Analysis and Semigroups*, IL, 1951.
6. M. G. Krein, D. P. Milman, *Studia math.*, **9**, 133 (1940).
7. M. M. Day, *Trans. Am. Math. Soc.*, **51**, 583 (1942).

Note: Figure translations are in progress. See original paper for figures.

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