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Abstract

Full Text

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PERIODIC, ALMOST-PERIODIC, AND BOUNDED SOLUTIONS OF THE DIFFERENTIAL EQUATION $\frac{dx}{dt} = f(t,x)$

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In this note conditions are given for the existence of a unique bounded solution $x^*(t)$ $(-\infty < t < +\infty)$, $|x^*(t)| \leq c$, of the equation

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

considered in a real Hilbert space H, and the behavior of the remaining solutions is studied. It is shown that this bounded solution is periodic (almost-periodic) if $f(t,x)$ is periodic (almost-periodic) in t. The precise formulation of the theorems is given below. Here and in what follows the norm of an element $x \in H$ is denoted by $|x|$. The results appear to be new also for the case of finite systems of ordinary differential equations.

By U we denote the Banach space of all continuous bounded functions $x(t)$ $(-\infty < t < +\infty)$ with values in H and norm $\|x(t)\|_0 = \sup_t |x(t)|$, and by W the Banach space of all $x(t) \in U$ possessing a continuous and bounded derivative, with norm $\|x(t)\| = \max\{\|x(t)\|_0, \|x'(t)\|_0\}$.

1. Let us first consider the linear differential equation

$$\frac{dx}{dt} = Ax + y(t), \tag{2}$$

in which A is a bounded self-adjoint invertible operator, and $y(t) \in U$. Let $H_- \oplus H_+ = H$ be the decomposition of H into invariant subspaces of the operator A corresponding to the negative and positive parts of its spectrum, and let A_-, A_+ be the parts of the operator A. Then

$$(A_- x, x) \leq -m_1(x,x) \quad (x \in H_-), \qquad (A_+ x, x) \geq m_2(x,x) \quad (x \in H_+), \tag{3}$$

where m_1, m_2 are certain positive numbers. Equation (2) has no more than one bounded solution.

Let us introduce the family of self-adjoint operators $\mathcal{K}(\sigma)$, defining it on the subspaces H_- and H_+ as follows:

$$\mathcal{K}(\sigma) = \begin{cases} \text{on } H_- : & e^{A_-\sigma}\ (\sigma \geq 0); \quad 0\ (\sigma < 0), \\ \text{on } H_+ : & -e^{A_+\sigma}\ (\sigma \leq 0); \quad 0\ (\sigma > 0). \end{cases} \tag{4}$$

It is not difficult to see that the operator

$$\mathcal{K}y(t) = \int_{-\infty}^{+\infty} \mathcal{K}(t-s)y(s)\,ds \tag{5}$$

acts from U into W, and $x^*(t) = \mathcal{K}y(t)$ is a bounded solution of equation (2). Equation (2) has no other bounded solutions. By direct calculation one can verify that the ope-

operator $\mathcal{K}$ takes an ω-periodic (almost-periodic) function $y(t) \in U$ into a function $\mathcal{K}y(t) \in W$ having the same property.

It is convenient for us to formulate these simple (and, apparently, known) facts in the form of a theorem.

Theorem 1. *Let A be a self-adjoint invertible operator, and let the function $y(t)$ be continuous and bounded. Then equation (2) has a unique bounded solution, which will be ω-periodic or almost-periodic if the function $y(t)$ has this property.*

The behavior of unbounded solutions of equation (2) is described by the following theorem, whose proof is easily obtained by using differential inequalities and inequalities (3).

Theorem 2. *If A is a negative-(positive-) definite self-adjoint operator, then every solution $x(t)$ of equation (2), as $t \to +\infty$ $(-\infty)$, tends exponentially to the bounded solution $x^*(t)$, and as $t \to -\infty$ $(+\infty)$ moves exponentially away from it.*

If A is a sign-changing operator, then for $x(0) - x^*(0) \in H_-$ the solution $x(t)$ tends exponentially to $x^*(t)$ as $t \to +\infty$ and moves exponentially away from it as $t \to -\infty$. The same statement, only in the reverse direction, holds if $x(0) - x^*(0) \in H_+$. And, finally, if $x(0) - x^*(0) \notin H_- \cup H_+$, then the solution $x(t)$ moves exponentially away from $x^*(t)$ as $t \to \pm\infty$.

Our aim is to carry over Theorems 1 and 2 to the nonlinear case.

2. In what follows we shall need two lemmas.

We shall agree to call a nonlinear operator acting in a Banach space **invertible** if it realizes a one-to-one and bicontinuous mapping of this Banach space onto itself.

By means of the contraction mapping principle one can prove

Lemma 1. *Let a family of operators $F_\mu(x)$, $0 \le \mu \le 1$, mapping the Banach space E into itself, be given and satisfy the conditions:*

1) *the operator $F_0(x)$ is invertible;*
2) *for all $\mu \in [0,1]$ and $x_1, x_2 \in E$ the inequality holds*

$$\|F_\mu(x_1) - F_\mu(x_2)\| \ge \varepsilon \|x_1 - x_2\|; \tag{6}$$

3) *for all $\mu, \mu_0 \in [0,1]$ and $x_1, x_2 \in E$ the inequality holds*

$$\|F_\mu(x_1) - F_\mu(x_2) - F_{\mu_0}(x_1) + F_{\mu_0}(x_2)\| \le \delta |\mu - \mu_0| \|x_1 - x_2\|, \tag{7}$$

where ε and δ are some positive constants.

Then the operator $F_\mu(x)$ is invertible for every $\mu \in [0,1]$, and, consequently, the equation $F_\mu(x) = 0$ has a unique solution for every $\mu \in [0,1]$.

By means of differential inequalities one proves

Lemma 2. *Let continuous bounded functions $a(t)$, $b(t)$, $u(t) \in H$ $(-\infty < t < +\infty)$ satisfy the conditions: $|a(t)| \le L|u(t)|$, $(a(t), Au(t)) \ge \varepsilon |u(t)|^2$ (A is a self-adjoint operator; $L, \varepsilon > 0$), and $u'(t) = a(t) + b(t)$. Then the estimate holds*

$$\|u(t)\|_0 < \alpha e^{L\alpha^2} \|b(t)\|_0, \tag{8}$$

where

$$\alpha = \left(\frac{\|A\|}{\varepsilon} + 1 + L \right).$$

3. We now give a theorem generalizing Theorem 1.

Theorem 3. *Let the operator $f(t,x)$ with values in H be continuous in t $(-\infty < t < +\infty)$ and satisfy the Lipschitz condition in x*

$$|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|. \tag{9}$$

Suppose the inequality

$$(f(t,x_1) - f(t,x_2),\ A(x_1 - x_2)) \ge \varepsilon |x_1 - x_2|^2$$
$$(-\infty < t < +\infty;\ x_1, x_2 \in H), \tag{10}$$

holds, where A is a self-adjoint operator and $\varepsilon > 0$ is constant.

Then, if $|f(t,0)|$ is bounded, equation (1) has a unique bounded solution. If, moreover, $f(t,x)$ is periodic in t ($f(t+\omega,x) \equiv f(t,x)$) or uniformly almost-periodic in t on every compact $K \subset H^*$, then this bounded solution will be, respectively, ω-periodic or almost-periodic.

Proof. Consider in the Banach space W the equation

$$x - P(x) = 0, \tag{11}$$

where the operator P is defined by the right-hand side of the equation

$$x(t) = \int_{-\infty}^{+\infty} \mathcal{K}(t-s)\{f(s, x(s)) - Ax(s)\}\, ds. \tag{12}$$

It is not difficult to see that the operator P acts in W, satisfies the Lipschitz condition

$$\|Px_1 - Px_2\| \leq \widetilde{L}\|x_1 - x_2\| \tag{13}$$

and that the bounded solutions of equation (1), and only they, are solutions of equation (11).

To prove the existence of a unique solution of equation (11), we proceed as follows. In the Banach space W consider the family of operators $F_\mu(x) = x - \mu P(x)$ and verify that all the conditions of Lemma 1 are fulfilled. Condition 1) is fulfilled, since $F_0(x) \equiv x$. From Lemma 2, after some calculations, we obtain that condition 2) is fulfilled with some constant ε, and, finally, from (13) we obtain that condition 3) is fulfilled with the constant $\widetilde{L}$. Consequently, by Lemma 1, equation (11) has a unique solution, i.e. equation (1) has a unique bounded solution $x^*(t)$.

Let us now note that, in order to prove that the solution $x^*(t)$ belongs to some subspace $\widetilde{W}$ of the space W (in what follows the role of $\widetilde{W}$ will be played by the subspace of ω-periodic functions or the subspace of almost-periodic functions), it is enough to show that the subspace $\widetilde{W}$ is invariant for the operator P. Indeed, in this case the family of operators $F_\mu(x) = x - \mu P(x)$, considered only on $\widetilde{W}$, satisfies all the conditions of Lemma 1, and since the operator P has only one fixed point in $W \supset \widetilde{W}$, this fixed point lies in $\widetilde{W}$.

Let $\widetilde{W} \subset W$ be the subspace consisting of all almost-periodic functions. Since the operator $\mathcal{K}$ carries every almost-periodic function $x(t) \in U$ into an almost-periodic function $\mathcal{K}x(t) \in \widetilde{W}$, in order to prove the invariance of $\widetilde{W}$ with respect to the operator P it is enough to show that the function $f(t, x(t))$ is almost-periodic if $x(t) \in \widetilde{W}$. It is easy to show that the set K of values of the almost-periodic function $x(t)$ is compact. Consider in the Banach space of continuous mappings of the compact set K into H the function $f(t) : f(t)x = f(t, x)$ $(x \in K)$. Since, by assumption, the operator $f(t, x)$ is continuous and almost-periodic in t uniformly with respect to $x \in K$, the function $f(t)$ is continuous and almost-periodic. Therefore, for a given $\varepsilon > 0$ one can indicate an $l > 0$ such that in every interval of length l there exists at least one value τ which is an ε-period simultaneously for both functions $x(t)$

* That is, for every $\varepsilon > 0$ one can indicate an $l > 0$ such that in each interval of length l there exists at least o

and $f(t)$: $|x(t+\tau)-x(t)| \leq \varepsilon$, $|f(t+\tau,x)-f(t,x)| \leq \varepsilon$ $(-\infty < t < +\infty,\ x \in K)$, whence it follows that

$$\begin{aligned} |f(t+\tau,x(t+\tau))-f(t,x(t))| &\leq |f(t+\tau,x(t+\tau))-f(t+\tau,x(t))| \\ &\quad + |f(t+\tau,x(t))-f(t,x(t))| \\ &\leq L|x(t+\tau)-x(t)|+\varepsilon \leq \varepsilon(1+L), \end{aligned}$$

i.e., the number τ is an $\varepsilon(1+L)$-period of the function $f(t,x(t))$. Consequently, the function $f(t,x(t))$ is almost periodic and $P\widetilde{W} \subseteq \widetilde{W}$.

In the case when the operator $f(t,x)$ is periodic in t: $f(t+\omega,x) \equiv f(t,x)$, the invariance of the subspace $\widetilde{W} \subseteq W$, consisting of all ω-periodic functions, is obvious.

The theorem is proved.

4. As a simple application of this theorem we give the following result for the second-order equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right), \tag{14}$$

considered in the Hilbert space H. If $f(t,x,y)$ is continuous and continuously differentiable with respect to x, y, and $f'_x(t,x,y)$ and $f'_y(t,x,y)$ are bounded and satisfy the inequalities

$$(f'_x(t,x,y)h,h) \geq m(h,h), \qquad |(f'_y(t,x,y)h,k)| \leq \delta|h|\,|k|, \tag{15}$$

where $m-(\delta/2)^2 > 0$, then, provided $|f(t,0,0)|$ is bounded, the assertions of Theorem 3 hold for equation (15).

5. Let the conditions of Theorem 3 be fulfilled. Then, if $|f(t,0)|$ is bounded, equation (1) has a unique bounded solution $x^*(t)$. The behavior of unbounded solutions of equation (1) is characterized by the following theorem, which generalizes Theorem 2.

Theorem 4. In the space H one can specify two manifolds $\mathfrak{M}_-$ and $\mathfrak{M}_+$ possessing the following properties. Both $\mathfrak{M}_-$ and $\mathfrak{M}_+$ are graphs of certain continuous mappings H_- (H_+) into H_+ (H_-). The manifold $\mathfrak{M}_-$ lies in the region $I_- : (Ax,x) \leq 0$, and the manifold $\mathfrak{M}_+$ in the region $I_+ : (Ax,x) \geq 0$ (in the case when $f(t,x) = A(t)x + y(t)$, both manifolds are subspaces whose direct sum is equal to H). If $x(0)-x^*(0) \in \mathfrak{M}_-$ $(\mathfrak{M}_+)$, then the difference $x(t)-x^*(t)$

tends exponentially to zero (to infinity) as $t \to +\infty$ and tends exponentially to infinity (to zero) as $t \to -\infty$. Finally, if $x(0) - x^*(0) \notin \mathfrak{M}_- \cup \mathfrak{M}_+$, then the difference $x(t) - x^*(t)$ tends exponentially to infinity as $t \to \pm\infty$.

Theorem 4, in the formulation of which we have restricted ourselves only to the case of a sign-changing operator A, generalizes a number of theorems on conditional stability from [4] to the case when equation (1) is considered as a whole.

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References

1. B. M. Levitan, *Almost-periodic functions*, Moscow, 1953.
2. B. P. Demidovich, *Matem. sbornik*, **40** (82), 73 (1956).
3. M. A. Krasnosel' skii, A. I. Perov, *DAN*, **123**, No. 2 (1958).
4. E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, IL, 1958.

Note: Figure translations are in progress. See original paper for figures.