MATHEMATICS

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1960

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Abstract

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ON THE TENSOR PRODUCT OF REPRESENTATIONS OF THE SUPPLEMENTARY SERIES OF THE PROPER LORENTZ GROUP

(Presented by Academician A. N. Kolmogorov, February 25, 1960)

In the author' s preceding note ([1]), the question was solved of into which irreducible representations the tensor product of two irreducible representations of the supplementary series of the proper Lorentz group decomposes. In the present note formulas are given which effect this decomposition; at the same time, the results of note ([1]) are obtained again by another method.*

1. **Functions** $\varphi(z_1, z_2)$. Let $x(g) \in K(\mathfrak{G})$; put

$$x(g) = x'(\delta z) = x'(\lambda, \zeta, z) \quad \text{for} \quad \delta = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix};$$

$$\varphi(z_1, z_2) = \int x\left(\lambda, \frac{1}{z_2 - z_1}, z_1\right) |z_2 - z_1|^{-2-\nu_2} |\lambda|^{-2-\nu_2+\nu_1}\, d\lambda. \tag{1}$$

I. If $x(g) \in K(\mathfrak{G})$, then $\varphi(z_1, z_2) \in \mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ and

$$(\varphi, \varphi) = \int h(g)\, x(gg')\, \overline{x(g')}\, dg\, dg', \tag{2}$$

where (φ, φ) denotes the square of the norm in $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$:

$$(\varphi, \varphi) = \int |z_1 - z_1'|^{-2+\nu_1} |z_2 - z_2'|^{-2+\nu_2} \varphi(z_1, z_2)\, \overline{\varphi(z_1', z_2')}\, dz_1\, dz_2\, dz_1'\, dz_2', \tag{3}$$

$$h(g) = |g_{21}|^{-2+\nu_1} |g_{12}|^{-2+\nu_2}.$$

The validity of this assertion may be verified by substituting in (3), in place of the function φ, its expression from (1), and by passing from integration with respect to z, ζ, λ to integration with respect to g.

Define in $K(\mathfrak{G})$ a scalar product by setting, for $x_1, x_2 \in K(\mathfrak{G})$,

$$(x_1, x_2) = \int h(g)\, x_1(gg')\, \overline{x_2(g')}\, dg\, dg', \tag{4}$$

and denote by $\mathcal{H}$ the Hilbert space which is the completion of $K(\mathfrak{G})$ with respect to this scalar product (strictly speaking, what is meant here is the completion of the factor space of $K(\mathfrak{G})$ by the subspace of those $x \in K(\mathfrak{G})$ for which $(x, x) = 0$). From (4) it follows immediately that the right-shift operator

$$B_{g_0} x(g) = x(gg_0)$$

defines a unitary representation $g \to B_g$ of the group $\mathfrak{G}$ in the space $\mathcal{H}$.

II. Formula (1) defines an isometric mapping of the space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ onto the space $\mathcal{H}$, under which the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ passes into the representation $g \to B_g$.

* Here we retain, for the most part, the notation of papers ([1–4]).

The first assertion follows from formula (2), and the second is obtained by direct verification from formula (1).

2. Decomposition of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ into irreducible representations for $\nu_1 + \nu_2 \leqslant 2$. Let first $\nu_1 + \nu_2 < 2$, and let $\chi(g) \in K(\mathfrak{G})$. According to the Plancherel formula for the group $\mathfrak{G}$,

$$\int \chi(gg')\overline{\chi(g')}\, dg' = \int K(\bar{z}g^{-1}, z', \chi)\alpha(\bar{z}g^{-1})K(z, z', \chi)\omega(\chi)\, dz\, dz'\, d\chi_0^+, \tag{5}$$

where, as usual,

$$K(z, z', \chi) = \int \chi(z^{-1}\delta\xi z')|\lambda|^{m-i\sigma-4}\lambda^{-m}\, d\lambda\, d\xi \quad \text{for} \quad \delta = \begin{Vmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{Vmatrix}, \tag{6}$$

$$z\bar{g} = \frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}, \qquad \alpha(g) = |g_{22}|^{-m+i\sigma-2}g_{22}^m, \qquad \chi(\lambda) = |\lambda|^{-m+i\sigma}\lambda^m,$$

$$\omega(\chi) = \frac{1}{(2\pi)^2}(m^2 + \sigma^2).$$

Substituting expression (5) into formula (2), changing the order of integration (which turns out to be legitimate when $\nu_1 + \nu_2 < 2$), and then making a change of integration variables, we arrive at the formula

$$(\varphi, \varphi) = \int |f(z, \chi)|^2 b(\chi)\, dz\, d\chi_0^+, \tag{7}$$

where

$$f(z, \chi) = \int K(z_1, z, \chi)|z_1|^{-1+\frac{\nu_1-\nu_2}{2}} \chi(\sqrt{z_1})\, dz_1, \tag{8}$$

$$b(\chi) = \frac{1}{2}\omega(\chi) \int |\xi - 1|^{-2+\nu_2} |\xi|^{\frac{\nu_1-\nu_2}{2}-1} \overline{\chi(\sqrt{\xi})}\, d\xi \times$$

$$\times \int |\eta - 1|^{-2+\nu_1} |\eta|^{\frac{\nu_2-\nu_1}{2}-1} \chi(\sqrt{\eta})\, d\eta. \tag{9}$$

The integrals in (9) are evaluated by applying a device analogous to Riesz' s device ([5]), after which the following formula is obtained for $b(\chi)$:

$$b(\chi) = \frac{1}{8} \frac{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}{\Gamma\left(1-\frac{\nu_1}{2}\right)\Gamma\left(1-\frac{\nu_2}{2}\right)} \left| \frac{\Gamma\left(\frac{m}{4} + i\frac{\sigma}{4} - \frac{\nu_1+\nu_2}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{m}{4} + i\frac{\sigma}{4} + \frac{\nu_1+\nu_2}{4} + \frac{1}{2}\right)} \right|^2 (m^2 + \sigma^2). \tag{10}$$

Next, combining formulas (1), (6), and (8) and carrying out, in essence, the same calculations as in ([4]), we conclude that

$$f(z, \chi) = \int \varphi(z_1, z_2) a(z_1, z_2, z, \nu_1, \nu_2, \chi)\, dz_1\, dz_2, \tag{11}$$

where

$$a(z_1, z_2, z, \nu_1, \nu_2, \chi) = |z_2 - z_1|^{\frac{m}{2} - i\frac{\sigma}{2} + \frac{\nu_1+\nu_2}{2} - 1} (z_2 - z_1)^{-\frac{m}{2}} \times$$

$$\times |z_2 - z|^{-\frac{m}{2} + i\frac{\sigma}{2} + \frac{\nu_2-\nu_1}{2} - 1} (z_2 - z)^{\frac{m}{2}} |z - z_1|^{-\frac{m}{2} + i\frac{\sigma}{2} + \frac{\nu_1-\nu_2}{2} - 1} (z - z_1)^{\frac{m}{2}} \tag{12}$$

for $\chi(\lambda) = |\lambda|^{-m+i\sigma}\lambda^m,\ \chi \in X_0^+$.

Formulas (7)–(12) remain valid also for $\nu_1 + \nu_2 = 2$, as can be verified by passing to the limit $\nu_1 + \nu_2 \to 2$ $(\nu_1 + \nu_2 < 2)$.

Under the transition from $x(g)$ to $x(gg_0)$, the function $\varphi(z_1, z_2)$ is transformed according to the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$, while $K(z_1, z, \chi)$—as a function of z, consequently (by virtue of (8)) also $f(z, \chi)$ as a function of z—is transformed according to the representation $\mathfrak{S}_\chi = \mathfrak{S}_{m,\sigma}$ of the principal series. Therefore, taking into account formulas (7), (10)–(12) and applying the continuous analogue of Schur' s lemma (see, for example, ([4]), p. 149), we arrive at the following theorem.

Theorem 1. Let $\mathfrak{H}$ be the Hilbert space of all measurable functions $f(x, \chi)$, $\chi \in X_0^+$, satisfying the condition

$$\int |f(z, \chi)|^2 b(\chi)\, dz\, d\chi_0^+ < \infty,$$

where $b(\chi)$ is defined by formula (10), with scalar product

$$(f_1, f_2) = \int f_1(z, \chi)\overline{f_2(z, \chi)}\, b(\chi)\, dz\, d\chi_0^+,$$

and let $\nu_1 + \nu_2 \leqslant 2$. Then for any function $\varphi(z_1, z_2) \in \mathfrak{H}'_{\nu_1} \times \mathfrak{H}'_{\nu_2}$ the integral in formula (11) converges in the sense of the norm in $\mathfrak{H}$, and the correspondence $\varphi \to f$ established by this formula defines an isometric mapping W of the space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ onto the space $\mathfrak{H}$. Under application to $\varphi(z_1, z_2)$ of the operator T_g of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$, the function $f(z, \chi) = W\varphi$ is transformed according to the representation $\mathfrak{S}_\chi = \mathfrak{S}_{m,\sigma}$ of the principal series. Consequently, the mapping W realizes the decomposition of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ into irreducible representations. In this decomposition there occur only representations $\mathfrak{S}_{m,\sigma}$ of the principal series, and precisely those, and only those, for which m is an even integer.*

3. Decomposition of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ into irreducible representations for $\nu_1 + \nu_2 > 2$. Let now $\nu_1 + \nu_2 > 2$. Then the change in the order of integration in the derivation of formula (7) is no longer legitimate, as a result of which this formula ceases to be valid. However, one can obtain a formula replacing formula (7) by applying to the latter the method of analytic continuation.

Denote by $L(Z \times Z)$ the totality of all infinitely differentiable finite functions $\varphi(z_1, z_2)$ whose supports do not contain the diagonal $z_1 = z_2$. Let $\varphi \in L(Z \times Z)$. From (3) it is immediately seen that (φ, φ) is an analytic function of ν_1, ν_2, regular for $0 < \operatorname{Re}\nu_1 < 2$, $0 < \operatorname{Re}\nu_2 < 2$. Let, further, $f(z, \chi, \nu_1, \nu_2)$ be the function defined by formula (8) (and hence by formulas (11), (12)), where $\chi(\lambda) = |\lambda|^{-m+i\sigma}\lambda^m$, m even and σ, in general, complex.

Put

$$b(\chi, \nu_1, \nu_2) = \frac{1}{8}\,\frac{\Gamma\bigl(\frac{\nu_1}{2}\bigr)\,\Gamma\bigl(\frac{\nu_2}{2}\bigr)}{\Gamma\bigl(1-\frac{\nu_1}{2}\bigr)\,\Gamma\bigl(1-\frac{\nu_2}{2}\bigr)}\,\frac{\Gamma\bigl(\frac{m}{4}+i\frac{\sigma}{4}-\frac{\nu_1+\nu_2}{4}+\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{m}{4}+i\frac{\sigma}{4}+\frac{\nu_1+\nu_2}{4}+\frac{1}{2}\bigr)}$$
$$\times\,\frac{\Gamma\bigl(\frac{m}{4}-i\frac{\sigma}{4}-\frac{\nu_1+\nu_2}{4}+\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{m}{4}-i\frac{\sigma}{4}+\frac{\nu_1+\nu_2}{2}+\frac{1}{2}\bigr)}\,(m^2+\sigma^2),$$

$$\psi(\chi, \nu_1, \nu_2) = \psi(m, \sigma, \nu_1, \nu_2) = \int f(z, \chi, \nu_1, \nu_2)\,\overline{f\bigl(z, \overline{\chi}^{-1}, \overline{\nu}_1, \overline{\nu}_2\bigr)}\, dz,$$

$$J_{\nu_1,\nu_2} = \int \psi(\chi,\nu_1,\nu_2)\, b(\chi,\nu_1,\nu_2)\, d\chi_0^+.$$

* The question of a formula for the inverse mapping W^{-1}, i.e. of reconstructing $\varphi(z_1, z_2)$ from $f(z,\chi)$, is mo

Formula (7) means that

$$(\varphi,\varphi) = J_{\nu_1,\nu_2} \quad \text{for } \operatorname{Im}\nu_1 = \operatorname{Im}\nu_2 = 0, \quad 0 \leqslant \nu_1,\ 0 \leqslant \nu_2,\ \nu_1+\nu_2 \leqslant 2. \tag{13}$$

Continuing analytically both parts of (13) from the domain $\operatorname{Re}(\nu_1+\nu_2) < 2$ to the domain $\operatorname{Re}(\nu_1+\nu_2) > 2$, and taking into account that

$$\int_0^\infty \frac{d\sigma}{\sigma^2 + (\nu_1+\nu_2-2)^2} = \begin{cases} \dfrac{\pi}{2(\nu_1+\nu_2-2)}, & \text{for } \operatorname{Re}(\nu_1+\nu_2) > 2, \\ -\dfrac{\pi}{2(\nu_1+\nu_2-2)}, & \text{for } \operatorname{Re}(\nu_1+\nu_2) < 2, \end{cases}$$

we obtain

$$(\varphi,\varphi) = \int |f(z,\chi)|^2 b(\chi)\, dy^+_{\chi_0} + \varkappa^2 (\hat f, \hat f)_{\nu_1+\nu_2-2} \quad \text{for } \nu_1 \leqslant 2,\ \nu_2 \leqslant 2,\ \nu_1+\nu_2 > 2,$$

where

$$\varkappa^2 = \frac{1}{\pi^2}\left[\frac{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})}{\Gamma(1-\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_2}{2})}\,\frac{\Gamma(1-\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1+\nu_2}{2}-1)}\left(1-\frac{\nu_1+\nu_2}{2}\right)\right]^2, \tag{14}$$

$$\hat f(z) = \int \varphi(z_1,z_2)\, |z_2 - z_1|^{\nu_1+\nu_2-2} |z_2 - z|^{-\nu_1} |z - z_1|^{-\nu_2}\, dz_1\, dz_2, \tag{15}$$

$(\hat f, \hat f)_{\nu_1+\nu_2-2}$ denotes the scalar product in $\mathfrak{H}_{\nu_1+\nu_2-2}$. Applying now arguments analogous to those of Sec. 2, we arrive at the following result.

Theorem 2. Let $\nu_1+\nu_2 > 2$, and let $\mathfrak{H}$ be defined as in Theorem 1. Let, further, $\mathfrak{H}$ denote the Hilbert space of all pairs

$$\tilde f = \{f(z,\chi), \hat f(z)\}, \quad f(z,\chi) \in \mathfrak{H}, \quad \hat f \in \mathfrak{H}_{\nu_1+\nu_2-2},$$

with scalar product

$$(\tilde f_1, \tilde f_2) = \int f_1(z,\chi)\overline{f_2(z,\chi)}\, b(\chi)\, dy^+_{\chi_0} + \varkappa^2 (\hat f, \hat f)_{\nu_1+\nu_2-2},$$

where $\varkappa^2$ is defined by formula (14). For any function $\varphi(z_1, z_2) \in \mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$, the integrals in formulas (11), (15) converge in the sense of the norm in $\mathfrak{H}$ and $\mathfrak{H}_{\nu_1+\nu_2-2}$, respectively, and these formulas define an isometric mapping $\dot{W}$ of the space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ onto the space $\mathfrak{H}$. When the operator T_g of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is applied to $\varphi(z_1, z_2)$, the functions $f(x, y)$, $\hat{f}(z)$ of the pair

$$\dot{W}\varphi = \{f(z, \chi), \hat{f}(z)\}$$

are transformed according to the representation $\mathfrak{S}_\chi = \mathfrak{S}_{m,\sigma}$ of the principal series and the representation $\mathfrak{D}_{\nu_1+\nu_2-2}$ of the supplementary series, respectively. Consequently, the mapping $\dot{W}$ effects the decomposition of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ into irreducible representations. This decomposition contains the representation $\mathfrak{D}_{\nu_1+\nu_2-2}$ of the supplementary series and precisely those representations $\mathfrak{S}_{m,\sigma}$ of the principal series for which m is an even number.

Received
8 II 1960

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Note: Figure translations are in progress. See original paper for figures.