Mathematics

1960

SovietRxiv

Abstract

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Mathematics

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The Operator of Fractional Differentiation and Powers of Elliptic Operators

(Presented by Academician S. L. Sobolev on 20 XI 1959)

Let Ω be a convex domain of n-dimensional Euclidean space. Let $f \in W_p^{(l)}(\Omega)$. If $lp > n$ and $0 < \alpha < \min\left[l - \frac{n}{p}, 1\right]$, then the fractional derivative $f^{(\alpha)}$ in the sense of ([3]) exists and is continuous with respect to (D, Q) in the domain $\Omega \times \Omega$. If, however, $lp \leq n$ and $0 < \alpha < l - \frac{n}{p} + \frac{n}{q}\left(p \leq q < \frac{np}{n-lp}\right)$, then the derivative $f^{(\alpha)}$ also exists and belongs to L_q with respect to (D, Q) in the domain $\Omega \times \Omega$ ([3]).

In both cases $f^{(\alpha)}$ is computed by the formula

$$f^{\alpha}(D,Q) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r (r-t)^{-\alpha-1}[f(Q)-f(D+et)]\left(\frac{t}{r}\right)^{n-1} dt + \frac{C_n^{(\alpha)}}{\Gamma(1-\alpha)}[f(Q)-f(D)]r^{-\alpha}. \tag{1}$$

We note that formula (1) remains valid also for $(D, Q) \in \overline{\Omega} \times \overline{\Omega}$. In this case, if $lp \leq n$, then for almost all D belonging to the boundary Γ of the domain Ω, $f^{(\alpha)}$ exists and, considered as a function of the single point Q, will belong to L_q.

Let D be a fixed point of the domain $\overline{\Omega}$. Consider the collection of all functions f from $W_p^{(l)}(\Omega)$ for which the function $f^{(\alpha)}(D, Q)$ is summable as a function of the point Q. If $lp > n$, then this collection coincides with all of $W_p^{(l)}(\Omega)$; if $lp \leq n$, then this collection is possibly a part of $W_p^{(l)}(\Omega)$. On the collection introduced we define the operator $\mathcal{D}^{\alpha} f(Q) = f^{(\alpha)}(D, Q)$, which we shall call the operator of fractional differentiation.

The purpose of the present paper is to compare the operator of fractional differentiation with powers of elliptic partial differential operators.

1. For the operator of fractional differentiation the following inequality holds.

Theorem 1. Let $f \in W_p^{(l)}(\Omega)$, $lp > n$, and $0 < \alpha < \min\left[l - \frac{n}{p}, 1\right]$. Then for all sufficiently small $\delta > 0$ the inequality

$$\|\mathcal{D}^\alpha f\|_{C(\Omega)} \leq \frac{K}{\delta^\nu}\|f\|_{L_p(\Omega)} + \delta^{1-\nu}\|f\|_{L_p^{(l)}(\Omega)} \tag{2}$$

holds, where

$$\nu = \frac{n}{pl} + \frac{\alpha}{l}.$$

The constant K does not depend on δ, f, or the point $D \in \overline{\Omega}$ with respect to which the operator $\mathcal{D}^\alpha$ is constructed.

The proof of this theorem is based on the use of the following inequalities $(1, 4)$:

$$|f(Q) - f(D)| \leq C_0 r^k \|f\|_{W_p^{(l)}(\Omega)}, \tag{3}$$

$$\|f\|_{C(\Omega)} \leq \frac{C_1}{\delta_0^{n/p}}\|f\|_{L_p(\Omega)} + C_2\delta_0^{l-n/p}\|f\|_{L_p^{(l)}(\Omega)}, \tag{4}$$

where

$$k = \min\left[l - \frac{n}{p}, 1\right] \qquad \|f\|^p_{W_p^{(l)}(\Omega)} = \|f\|^p_{L_p(\Omega)} + \|f\|^p_{L_p^{(l)}(\Omega)},$$

δ_0 is any sufficiently small positive number, and the constants C_1 and C_2 do not depend on δ_0 or f.

2. If $f \in W_p^{(l)}(\Omega)$ and $lp \leq n$, then, by the embedding theorem (1), the function f from $W_p^{(l)}(\Omega)$ has summable values on the boundary Γ of the domain Ω. Therefore the set of those $f \in W_p^{(l)}(\Omega)$ for which $f|_\Gamma = 0$ forms a subspace of the space $W_p^{(l)}(\Omega)$, which we shall denote by $\overset{0}{W}{}_p^{(l)}(\Omega)$.

We shall assume that the point D with respect to which the operator $\mathcal{D}^\alpha$ is constructed belongs to Γ.

Theorem 2. Let $f \in \overset{0}{W}{}_p^{(l)}(\Omega)$, $0 < \alpha < l - \dfrac{n}{p} + \dfrac{n}{q}$, $q > p$. Then, for all sufficiently small $\delta > 0$, the inequality

$$\|\mathcal{D}^\alpha f\|_{L_q(\Omega)} \leq \frac{K}{\delta^\nu}\|f\|_{L_p(\Omega)} + \delta^{1-\nu}\|f\|_{L_p^{(l)}(\Omega)}, \tag{5}$$

holds, where

$$\nu = \frac{n}{l}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{\alpha + \beta}{l}.$$

The constant K does not depend on δ, f, or the point $D \in \Gamma$ with respect to which the operator $\mathcal{D}^\alpha$ is constructed; β is an arbitrarily small fixed positive number.

In proving this inequality we use the inequality: for $u \geq 0$,

$$\int_{\omega'} d\chi \int_0^{d(e)-u} |f(Q+eu)-f(Q)|^q r^{n-1}\, dr \leq C_1 u^k \|f\|_{W_p^{(l)}(\Omega)}, \tag{6}$$

where $f \in W_p^{(l)}(\Omega)$ $(lp \leq n)$; $k = l - \dfrac{n}{p} + \dfrac{n}{q}$; e is the unit vector having the direction from D to Q; r is the distance between the points D and Q; $d(e)$ is the length of the segment of the ray issuing from the point D in the direction e within the domain $\overline{\Omega}$; $d\chi$ is the surface element of the unit sphere in n-dimensional space; ω' is the corresponding part of this surface. In addition, we use inequality (4)

$$\left\|\frac{f}{r^{\alpha'}}\right\|_{L_q(\Omega)} \leq \frac{K_1}{\delta_0^{\frac{n}{p}-\frac{n}{q}+\alpha'}}\|f\|_{L_p(\Omega)} + K_2\delta_0^{l-\frac{n}{p}+\frac{n}{q}-\alpha'}\|f\|_{L_p^{(l)}(\Omega)}, \tag{7}$$

where K_1 and K_2 do not depend on δ_0, f, or the point $D \in \overline{\Omega}$; δ_0 is a sufficiently small positive number.

We note that in the case where the order of the derivative is a positive integer, the inequalities corresponding to inequalities (2) and (5) are known ([4]).

3. Let us now turn to a comparison of the fractional differentiation operator $\mathcal{D}^\alpha$ with differential operators in partial derivatives. Consider, for example, the elliptic operator of second order

$$\mathcal{L}f = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,k}(x)\frac{\partial f}{\partial x_k}\right), \tag{8}$$

defined on $\overset{0}{W}{}_2^2(\Omega)$. Under the known assumptions the operator $\mathcal{L}$ will be a positive-definite self-adjoint operator, for which the Bernstein–Ladyzhenskaya inequality is valid:

$$\|f\|_{\overset{0}{W}{}_2^2(\Omega)} \leq C\|\mathcal{L}f\|_{L_2(\Omega)}. \tag{9}$$

From inequalities (5) and (9) we obtain

$$\|\mathcal{D}^\alpha f\|_{L_2(\Omega)} \leq \frac{\widetilde{K}}{\delta^\nu}\|f\|_{L_2(\Omega)} + \delta^{\nu'}\|\mathcal{L}f\|_{L_2(\Omega)} \tag{10}$$

with the corresponding value of ν and the constant $\widetilde{K}$.

From this inequality, by virtue of the theorem of S. G. Krein and P. E. Sobolevskii ([2]), it follows that

Theorem 3. *The fractional differentiation operator $\mathcal{D}^\alpha$ is an operator of fractional order not higher than $\alpha/2$ with respect to the operator $\mathcal{L}$; i.e., on $\overset{0}{W}{}_2^2(\Omega)$, for $\alpha/2 < \gamma < 1$, the inequality*

$$\|\mathcal{D}^\alpha f\|_{L_2(\Omega)} \leq C\|\mathcal{L}^\gamma f\|_{L_2(\Omega)} \tag{11}$$

holds. Here C does not depend on $D \in \Gamma$.

Theorem 4. *Let $2\gamma - n/2 \leq \alpha < 2\gamma$. Then the operator $\mathcal{D}^\alpha \mathcal{L}^{-\gamma}$ is a bounded operator acting from $L_2(\Omega)$ into $L_q(\Omega)$, where*

$$\frac{1}{q} > \frac{1}{2} - \frac{2\gamma - \alpha}{n}.$$

The fact that the fractional differentiation operator $\mathcal{D}^\alpha$ is an operator of fractional order with respect to the operator $\mathcal{L}$ makes it possible to consider differential equations which contain, in addition to partial derivatives of the unknown function, its fractional derivatives.

Consider the following problem. Find a solution of the equation

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,k}(x)\frac{\partial f}{\partial x_k}\right) + p(x)\mathcal{D}^\alpha f + q(x)f = \varphi(x), \tag{12}$$

where $p(x)$ and $q(x)$ are bounded functions satisfying the boundary condition $f|_\Gamma = 0$.

For this problem the following is valid.

Theorem 5. *The spectrum of the operator standing in the left-hand side of the equation will be located in some half-plane* $\operatorname{Re}\lambda \geq \omega$.

If $q(x) > A$, where A is a sufficiently large constant, then problem (12) has a unique solution $f \in \overset{0}{W}{}_2^2(\Omega)$.

Similarly, one may consider equations of parabolic type.

Consider the equation

$$\frac{\partial f}{\partial t} = \mathcal{L}f + p(x)\mathcal{D}^\alpha f + q(x)f$$

with the boundary condition $f|_\Gamma = 0$ and the initial condition $f(0,x) = v(x) \in L_2(\Omega)$.

It can be proved that a solution of the equation exists in $\overset{0}{W}{}_2^2(\Omega)$ and is analytic in t inside some angle $|\arg t| < \varphi_0$ (φ_0 does not depend on t).

Using the results of ([6,5,7]), one can obtain analogous results also in the space $L_p(\Omega)$.

One may also consider the case of equations with a strongly elliptic operator of order $2m$.

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Received
18 XI 1959

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Note: Figure translations are in progress. See original paper for figures.