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Abstract

Full Text

MATHEMATICS

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ON ORTHOGONAL POLYNOMIALS ON SEVERAL INTERVALS

(Presented by Academician S. N. Bernstein, 18 IV 1960)

1. In the study of the asymptotic properties of orthogonal polynomials, due to S. N. Bernstein and G. Szegő, it is assumed that the set of points of the interval of orthogonality at which the weight vanishes has measure zero. In some cases one even has to assume that the number of these points is finite. In the investigations mentioned, the starting point is the classical systems of orthogonal polynomials, whose weights inside the interval of orthogonality do not vanish at all.

In the present note we shall prove a proposition which pertains to the case where the weight is equal to zero on entire intervals, and which also makes it possible to include this case within the scope of contemporary general investigations.

Assume that the weight is different from zero only at points belonging to the intervals

$$(E)\ [-1,\alpha_1],\ [\beta_1,\alpha_2],\dots,[\beta_\rho,1] \qquad (-1<\alpha_1<\beta_1<\dots<\alpha_\rho<\beta_\rho<1),$$

and has one of the following two forms:

$$\frac{S(x)}{\sqrt{-R(x)}}\frac{1}{t(x)}, \qquad \frac{\sqrt{-R(x)}}{S(x)}\frac{1}{t(x)} \qquad (x\in E), \tag{1}$$

where $t(x)$ is a continuous positive function of $x \in E$, and $S(x)$, $R(x)$ are defined by the formulas

$$S(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_\rho), \qquad R(x)=(x+1)(x-\alpha_1)(x-\beta_1)\cdots$$

$$\cdots(x-\beta_\rho)(x-1).$$

The ratio $S(x) : \sqrt{-R(x)}$, which enters into the weight, is regarded as positive inside E. The orthogonal polynomials with respect to the weights (1) will be denoted respectively by

$$T_n(x;t) = x^n + \cdots, \qquad U_n(x;t) = x^n + \cdots.$$

We shall have to consider the radical $\sqrt{R(z)}$ also in the domain $\mathfrak{G}$, which is obtained from the complex z-plane by means of cuts along the intervals E. We shall take that branch of the radical which is positive at some point $z = x > 1$. This branch is single-valued, and, by virtue of the convention made above, the equality holds

$$\sqrt{R(x \pm i0)} = \pm i\sqrt{-R(x)} \qquad (x \in E). \tag{2}$$

Finally, introduce the two-sheeted Riemann surface $\mathfrak{F}$, whose transition lines are the segments of the system (E), and whose branch points are the endpoints of these segments. One sheet of the surface $\mathfrak{F}$ will be $\mathfrak{G}$; the second sheet we shall call $\mathfrak{G}'$, and if c is some point of the sheet $\mathfrak{G}$, then the corresponding

the point of the sheet $\mathfrak{G}'$ will be denoted by c'. Thus, $\sqrt{R(c')} = -\sqrt{R(c)}$ and, in particular,

$$\lim_{z\to\infty'} \frac{\sqrt{R(z)}}{z^{p+1}} = -\lim_{z\to\infty} \frac{\sqrt{R(z)}}{z^{p+1}} = -1 \quad (z \in \mathfrak{F}).$$

Lemma. Let $P(x)$ and $Q(x)$ be two polynomials, both of even degree and both positive for $x \in E$. Let $a_1, a_2, \dots, a_p$ be the roots of the first, and $b_1, b_2, \dots, b_q$ $(q > p,\ q \geq p)$ the roots of the second polynomial (the roots being written with multiplicity). Suppose, finally, that the equalities

$$\int_E \frac{S(x)}{\sqrt{-R(x)}} \frac{\ln P(x)}{x - \alpha_j}\, dx = \int_E \frac{S(x)}{\sqrt{-R(x)}} \frac{\ln Q(x)}{x - \alpha_j}\, dx \quad (j = 1, 2, \dots, p). \tag{3}$$

hold. In that case there exists a rational function $f(z, \sqrt{R(z)})$, whose only zeros on the surface $\mathfrak{F}$ are the points $b_1, b_2, \dots, b_q$, and whose only poles are the points $a_1, a_2, \dots, a_p$ and the point ∞, at which the pole has multiplicity $q - p$. (Replacing the function $f(z, \sqrt{R(z)})$ by $f(z, -\sqrt{R(z)})$ corresponds to replacing the points b_i, a_k, ∞ by the points b_i', a_k', ∞'.)

2. Let $P(x)$ be a polynomial of even degree p, positive for $x \in E$, and let a_k $(k = 1, 2, \dots, p)$ be its roots. We shall prove that, in order to find the orthogonal polynomials $T_n(x; P)$, $U_{n-1}(x; P)$ for $n \geq p$, it is enough to solve the following problem:

Problem. Construct a rational function $p(z, \sqrt{R(z)})$ having a pole of order n at the point ∞', p poles of the first order α_i $(j = 1, 2, \dots, p)$, a zero of multiplicity $n - p$ at the point ∞, and p zeros a_k $(k = 1, 2, \dots, p)$.

The correctly counted number of poles of the required rational function is $n+p$, while the correctly counted number of zeros is n. The function $p(z, \sqrt{R(z)})$ is determined by these conditions up to an arbitrary constant factor and will have another p zeros, which can no longer be prescribed in advance. It will be determined completely if we require, for example, that

$$\lim_{z\to\infty'} \frac{p(z, \sqrt{R(z)})}{z^n} = \lim_{z\to\infty} \frac{p(z, -\sqrt{R(z)})}{z^n} = 2.$$

Thus, suppose that the function $p(z, \sqrt{R(z)})$ has been constructed. Every rational function of z and $\sqrt{R(z)}$ can be represented in the form

$$\frac{A(z) + B(z)\sqrt{R(z)}}{C(z)},$$

where $A(z), B(z), C(z)$ are polynomials in z. In the case considered by us, as is easy to verify,

$$p(z, \sqrt{R(z)}) = A_n(z) - \frac{\sqrt{R(z)}}{S(z)} B_{n-1}(z), \tag{4}$$

where $A_n(z)$ is a polynomial of degree n, and $B_{n-1}(z)$ is a polynomial of degree $n-1$, both with leading coefficients equal to 1. We shall now prove that $A_n(z) = T_n(z; P)$ and $B_{n-1}(z) = U_{n-1}(z; P)$. To this end, denote by L a system of nonintersecting ovals surrounding the intervals belonging to (E). These ovals are assumed to be such that all the points a_k lie inside the infinite domain $\mathfrak{D} \subset \mathfrak{G}$ bounded by them. Consider the integral

$$I_m = \int_L p(z, \sqrt{R(z)}) \frac{S(z)}{\sqrt{R(z)}} \frac{z^m\, dz}{P(z)} \quad (m = 0, 1, 2, \dots, n-1).$$

Its subintegral function is regular everywhere in $\mathfrak{D}$, and in a neighborhood of the point ∞ is equal to $O(z^{m-n-1})$. Therefore $I_m = 0$ $(m = 0, 1, 2, \dots, n-1)$. Replacing the function $p(z, \sqrt{R(z)})$ by its expression (4), we find that

$$\int_L A_n(z) \frac{S(z)}{\sqrt{R(z)}} \frac{z^m\, dz}{P(z)} = 0 \qquad (m = 0, 1, 2, \dots, n-1),$$

or, taking (2) into account:

$$\int_E A_n(x)\frac{S(x)}{\sqrt{-R(x)}}\frac{x^m\,dx}{P(x)} = 0 \qquad (m = 0, 1, 2, \dots, n-1).$$

Thus it has been proved that $A_n(z) = T_n(z; P)$. Similarly, considering the integral

$$\int_L p(z, \sqrt{R(z)})\frac{z^m\,dz}{P(z)} \qquad (m = 0, 1, 2, \dots, n-2),$$

we find that $B_{n-1}(z) = U_{n-1}(z; P)$.

3. Let us now take a second polynomial, denote it by $Q(x)$, of even degree and positive for $x \in E$. Suppose that it satisfies conditions (3), and let its degree q be greater than p. To find the polynomials $T_n(z; Q)$, $U_{n-1}(z; Q)$ for $n > q$, one should repeat the constructions of the preceding item as applied to $Q(x)$. They lead to the function

$$q(z, \sqrt{R(z)}) = T_n(z; Q) - \frac{\sqrt{R(z)}}{S(z)}U_{n-1}(z; Q).$$

However, thanks to condition (3) and the lemma, one can obtain the function $q(z, \sqrt{R(z)})$ from the function $p(z, \sqrt{R(z)})$ by the formula

$$q(z, \sqrt{R(z)}) = p(z, \sqrt{R(z)})\Phi(z, \sqrt{R(z)}),$$

where the function $\Phi(z, \sqrt{R(z)})$ is completely determined by the following conditions: it has zeros and poles only on the sheet $\mathfrak{G}$, with its zeros coinciding with the zeros of the polynomial $Q(z)$, while its poles are the zeros of the polynomial $P(z)$ and, in addition, the point ∞, at which $\Phi(z, \sqrt{R(z)})$ has a pole of order $q-p$. The function $\Phi(z, -\sqrt{R(z)})$ has analogous properties, with the difference that all its zeros and poles lie on the sheet $\mathfrak{G}'$. Therefore

$$\Phi(z, \sqrt{R(z)})\Phi(z, -\sqrt{R(z)}) = C\frac{Q(z)}{P(z)},$$

where C is a constant. The quantities $\Phi(x, \sqrt{R(x)})$, $\Phi(x, -\sqrt{R(x)})$ have complex-conjugate values for $x \in E$. Therefore the function $\varphi(z) = \Phi(z, -\sqrt{R(z)})$ has the following properties: $\varphi(z)$ is single-valued and regular in the domain $\mathfrak{G}$; $\varphi(\infty) = 1$; for $x \in E$ the equality

$$|\varphi(x)| = C_1\sqrt{\frac{Q(x)}{P(x)}},$$

holds, where C_1 is a constant. These properties show that the function $\varphi(z)$ can be obtained in the form of an integral as the result of solving a certain boundary-value problem, namely:

$$\varphi(z) = G\left[\frac{Q(x)}{P(x)}\right] \cdot A\left[z; \frac{Q(x)}{P(x)}\right],$$

where

$$G[f(x)] = \exp\left\{-\frac{1}{2\pi}\int_E \frac{S(x)}{\sqrt{-R(x)}} \ln f(x)\, dx\right\},$$

$$A[z; f(x)] = \exp\left\{\frac{1}{2\pi}\int_E \frac{\sqrt{R(z)}}{\sqrt{-R(x)}} \frac{\ln f(x)}{z-x}\, dx\right\}.$$

From what has been said there follows the following

Theorem. *If the polynomials $P(x)$ and $Q(x)$ satisfy the conditions of the lemma and $n \geq q$, then for $z \in \mathfrak{G}$*

$$\frac{T_n(z;Q) + \dfrac{\sqrt{R(z)}}{S(z)} U_{n-1}(z;Q)}{T_n(z;P) + \dfrac{\sqrt{R(z)}}{S(z)} U_{n-1}(z;P)} = G\left[\frac{Q(x)}{P(x)}\right] A\left[z; \frac{Q(x)}{P(x)}\right]. \tag{5}$$

This is precisely the proposition that we wished to prove. With its aid, orthogonal polynomials for one weight are expressed through orthogonal polynomials for another weight.

If $t(x)$ $(x \in E)$ is an arbitrary continuous positive function, then for any $\varepsilon > 0$ there will be a polynomial $Q(x)$ of even degree such that

$$\frac{1}{1+\varepsilon} < \frac{t(x)}{Q(x)} < 1 + \varepsilon \qquad (x \in E),$$

$$\int_E \frac{S(x)}{\sqrt{-R(x)}} \ln t(x) \frac{dx}{x-\alpha_j} = \int_E \frac{S(x)}{\sqrt{-R(x)}} \ln Q(x) \frac{dx}{x-\alpha_j} \qquad (j = 1, 2, \dots, \rho).$$

Using this simple fact, one can obtain asymptotic formulas for large n for the polynomials $T_n(z;t)$, $U_{n-1}(z;t)$, expressing these polynomials through the "simpler" polynomials $T_n(z;P)$, $U_{n-1}(z;P)$.

The case in which the set E consists of a single interval $[-1, 1]$ is included as a limiting case. In this case the conditions (3) disappear, and one may take $P(z) \equiv 1$. Then

$$T_n(z;1) + \sqrt{z^2-1}\,U_{n-1}(z;1) = \frac{\left(z+\sqrt{z^2-1}\right)^n}{2^{n-1}},$$

and formula (5) leads to the relation

$$\frac{T_n(z;Q) + \sqrt{z^2-1}\,U_{n-1}(z;Q)}{T_m(z;Q) + \sqrt{z^2-1}\,U_{m-1}(z;Q)} = \left(\frac{z+\sqrt{z^2-1}}{2}\right)^{n-m} \qquad (n \geq m \geq q),$$

which is found in S. N. Bernstein (see (1), [42]).

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[2] G. Szegő, *Orthogonal Polynomials*, N. Y., 1939.

Note: Figure translations are in progress. See original paper for figures.