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Abstract

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A LOWER BOUND FOR THE RATE OF CONVERGENCE OF DISTRIBUTIONS OF SUMS TO THE CLASS OF INFINITELY DIVISIBLE LAWS

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Let:

1) $\mathfrak{G}$ be the class of infinitely divisible laws;

2) $\mathfrak{G}_L$ be the class of $G \in \mathfrak{G}$ such that the logarithm of their characteristic function $\ln g(t)$ can be represented in the form

$$\ln g(t) = i\gamma t + \int_{-L}^{L} \left(e^{itu} - 1 - itu\right) \frac{dK(u)}{u^2},$$

where $\gamma^2 \geq 0$, $K(u)$ is a nondecreasing function of bounded variation;

3) C_k $(k = 1, 2, ...)$ be absolute constants > 0 and $< \infty$;

4) $F^n(x) = F(x) * ... * F(x)$ be the distribution function of the sum of n independent random variables, each of which has distribution function $F(x)$;

5) $\rho(F, G) = \sup_x |F(x) - G(x)|$;

6) $\rho(F, \mathfrak{G}) = \inf_{G \in \mathfrak{G}} \rho(F, G)$; $\rho(F, \mathfrak{G}_L) = \inf_{G \in \mathfrak{G} L} \rho(F, G)$;

7)

$$F_p(x) = \begin{cases} 0, & x \leq 0, \\ 1 - p, & 0 < x \leq 1, \\ 1, & x > 1; \end{cases}$$

8) $F_p^*(x) = F_p(x) * (1 - F_p(-x + 0))$ be the distribution function of $\eta = \xi_1 - \xi_2$, where ξ_i $(i = 1, 2)$ are independent and each of them has distribution function $F_p(x)$;

9) $\Psi(n) = \sup_F \rho(F^n, \mathfrak{G})$;

10) $\Psi_1(n) = \sup_{p \leq 1} \rho(F_p^n, \mathfrak{G})$, $\Psi_2(n) = \sup_{p \leq 1} \rho(F_p^{*n}, \mathfrak{G})$.

A. N. Kolmogorov proved (^{1}) that

$$\Psi(n) < C_1 n^{-1/5}.$$

Yu. V. Prokhorov succeeded in strengthening this result (^{2}). He showed that

$$C_2 (n \ln n)^{-1} < \Psi(n \ln n) < C_3 n^{-1/3} (\ln n)^2.$$

A lower bound for $\Psi(n)$ was obtained by Yu. V. Prokhorov by estimating $\Psi_2(n)$ from below. On the other hand, in the author' s work (^{3}) it was proved that

$$\Psi_1(n) < C_4 n^{-2/3}, \qquad \Psi_2(n) < C_5 n^{-1}.$$

It turns out that $\Psi_1(n)$, up to a logarithmic factor, decreases as $n^{-2/3}$. More precisely, the following is true:

Theorem 1. *For any function $u(n) \to 0$ $(n \to \infty)$, there exists an n_0 such that, for $n > n_0$,*

$$\Psi_1(n) > n^{-2/3} (\ln n)^{-3/2} u(n).$$

In view of the fact that $\Psi(n) \geq \Psi_1(n)$, this theorem immediately implies:

Theorem 2.

$$\Psi(n) > C_6 n^{-2/3} (\ln n)^{-4}.$$

It remains to briefly outline the proof of Theorem 1.

In what follows:

1) $p = n^{-2/3}$;

2) $\alpha = \alpha(n)$ is an arbitrary function satisfying the following two conditions: a) $\alpha < \ln n$, b) $\alpha \to \infty$ $(n \to \infty)$;

3) $L = n^{1/6} (\ln n)^{1/2} \alpha$;

4) $\varkappa_k$ are the semi-invariants of G;

5) $G^*(x) = G(x) * (1 - G(-x + 0))$.

We shall say that G satisfies condition (A_n) if:

1) $\rho(F_p^n, G) < C_4 p$;

2) $G \in \mathfrak{G}_L$.

Since $\rho(F_p^n, \mathfrak{G}) \leq \Psi_1(n)$, to prove Theorem 1 it is enough to show that there exist $k < \infty$ and $n_0 = n_0(\alpha)$ such that, for $n > n_0$,

$$\rho(F_p^n, \mathfrak{G}) > p(\ln n)^{-3/2}\alpha^{-k}. \tag{1}$$

The following lemmas are proved successively:

Lemma 1. *There exists an n_1 such that, for $n > n_1$,*

$$\rho(F_p^n, \mathfrak{G}_L) < 20\rho(F_p^n, \mathfrak{G}). \tag{2}$$

Lemma 2. *There exists an n_2 such that, if for $n > n_2$ the condition (A_n) holds for G, then for $x > L\ln n\alpha$:*

a)

$$G^*(-x) < \exp\left\{-\frac{x}{2L}\right\};$$

b)

$$\max\{G(\gamma - x),\, 1 - G(\gamma + x)\} < \exp\left\{-\frac{x}{2L}\right\}, \quad \text{where } \gamma = \int_{-\infty}^{\infty} x\, dG(x).$$

In the proof of this lemma one first estimates the variance of G^*, and then, using an exponential strengthening of Chebyshev' s inequality, the resulting estimate is used to prove a) and b).

Lemma 3. *There exists an n_3 such that, if for $n > n_3$ condition (A_n) holds for G, then $g(t)$, the characteristic function of G, can be represented in the form*

$$g(t) = \exp\left\{i\gamma t + \sum (e^{itk_i} - 1)q_k + \int_{-L}^{L} (e^{itu} - 1)\, dK'(u)\right\}; \tag{3}$$

where γ is an integer, $q_k \geq 0$, $K(u)$ is a nondecreasing function, and moreover

$$\delta = \int_{-L}^{L} dK(u) < n^{-1/2}(\ln n)^5. \tag{4}$$

Condition (A_n) imposes a fairly severe restriction on the increment of the function G^* at noninteger points; namely, by Lemma 2, for sufficiently large n

$$\sum_{k=-\infty}^{\infty} \{G^*(k+1) - G^*(k+0)\} \leq 2C_4\rho L\ln n\alpha + 2G^*(-L\ln n\alpha) \leq n^{-1/2}(\ln n)^4. \tag{5}$$

Let G_1 be the infinitely divisible law corresponding to the characteristic function

$$g_1(t) = \exp\left\{2\int_{-L}^{L} (\cos tu - 1)\, dK(u)\right\},$$

where $u^2K(u)$ is a nondecreasing function of bounded variation, continuous at the integral points. In the proof of the lemma it is shown that there exist C_7 and n_4 such that, for $n > n_4$, the increment at nonintegral points of G_1 is greater than $C_7 \min(\delta, 1)$. Hence, by virtue of (5), (4) follows. It is clear that, when (4) is satisfied, for sufficiently large n, γ must be an integer.

Lemma 4. There exists n_5 such that if, for $n > n_5$, G satisfies (A_n), then

$$\left|\int_{-\infty}^{\infty} x\, dG(x) - np\right| < n^{-1/2}(\ln n)^5. \tag{6}$$

This lemma is proved with the aid of integration by parts of

$$\frac{1}{t}\int_{-\infty}^{\infty} e^{itx}\, d(F_p^n(x) - G(x))$$

and the subsequent passage to the limit as $t \to 0$.

Lemma 5. There exists n_6 such that if, for $n > n_6$, G satisfies (A_n) and

$$|x_2 - np(1-p)| \le \frac{1}{2}np^2, \tag{7}$$

then

$$|x_4 - np| > \frac{1}{5}. \tag{8}$$

The proof of this lemma is based on the fact that, if (8) does not hold, then, under conditions (6), (7), for an infinitely divisible law whose characteristic function has the form (3), (4), $|x_3 - np| > 1/4$. But for large values of n this is incompatible with the assumption that (A_n) is satisfied.

It follows from Lemma 5 that, if for $n > n_6$ G satisfies (A_n), then only 3 cases are possible:

1) $x_6 > 2np$;
2) $x_6 \le 2np,\ |x_4 - np| > 1/5$;
3) $x_6 \le 2np,\ |x_4 - np| \le 1/5,\ |x_2 - np(1-p)| > \frac{1}{2}np^2$.

Applying Parseval' s formula to the difference $F_p^{*n} - G^*$ and using Lemma 2, it is shown that there exist n_7 and $k < \infty$ such that, for $n > n_7$, in each of these cases,

$$\rho(F_p^n, G) > \frac{1}{20} n^{-2/3} (\ln n)^{-3/2} \alpha^{-k}. \tag{9}$$

In view of Lemma 1, (1) follows from (9).

In conclusion I take the opportunity to sincerely thank A. N. Kolmogorov and Yu. V. Prokhorov for their attention to the present work.

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Note: Figure translations are in progress. See original paper for figures.