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Abstract

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ON THE EMBEDDING OF NUCLEAR SPACES IN THE SPACE OF ALL INFINITELY DIFFERENTIABLE FUNCTIONS ON THE LINE

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In Grothendieck' s memoir ((*1*), part 2, pp. 68, 139) the following problem is formulated.

Can every metrizable nuclear space be isomorphically embedded in the space of all infinitely differentiable functions on the line?

The present paper contains a partial answer to this question.

1. **Definition 1.** Let X and Y be linear topological spaces. The space Y is called **isomorphically embeddable** in X if there exists an isomorphic (i.e., linear, homeomorphic) mapping of the space Y onto some subspace Y_1 of the space X. The spaces X and Y are **isomorphic** if $Y_1 = X$.

Definition 2 ((*1*), part 2, p. 34; (*2*)). A locally convex space X is called **nuclear** if, for every continuous seminorm $\|\cdot\|_\alpha$ defined on X, there exist: a continuous seminorm $\|\cdot\|_{\alpha'}$, vectors $(x_n^{(\alpha)})$, and linear functionals $(f_n^{(\alpha)})$ such that:

$$\text{for every } x \in X \qquad \lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} f_i^{(\alpha)}(x) x_i^{\alpha} \right\|_\alpha = 0; \tag{1}$$

$$\sum_{n=1}^{\infty} \sup\{f_n^{(\alpha)}(x) : \|x\|_{\alpha'} \le 1\} < +\infty, \qquad \left\|x_n^{(\alpha)}\right\|_\alpha = 1 \quad (n = 1, 2, \ldots). \tag{2}$$

In what follows we shall consider only **metrizable locally convex** spaces, i.e. such spaces whose topology is defined by a countable family of seminorms $\|\cdot\|_\alpha$ $(\alpha = 1, 2, \ldots)$.

By $\mathcal{E}(R)$ we denote the space of all infinitely differentiable functions $x = x(t)$ defined on $R = (-\infty, \infty)$, with the topology defined by the seminorms

$$\|x\|_\alpha = \sup_{|t|\le\alpha} \left[|x(t)| + |x^{(1)}(t)| + \cdots + |x^{(\alpha)}(t)| \right];$$

by $\mathcal{E}(I)$, the space of all infinitely differentiable functions on the interval $I = [-1, 1]$ with seminorms

$$\|x\|_\alpha = \sup_{t \in I} |x^{(\alpha)}(t)|;$$

by $\mathcal{E}(K)$, the subspace of the space $\mathcal{E}(R)$ consisting of all periodic functions with period 2π (the space of functions on the circle K).

Let $a_\alpha^{(n)}$ $(\alpha, n = 1, 2, ...)$ be nonnegative real numbers such that $a_\alpha^{(n)} \leq a_{\alpha+1}^{(n)}$, $\sup_\alpha a_\alpha^{(n)} > 0$. By $l(a_\alpha^{(n)})$ we denote the **Köthe space** (*3*) generated by the matrix $(a_\alpha^{(n)})$, i.e. the space of all numerical sequences $\xi = (\xi_n)$ for which

$$\|\xi\|_\alpha = \sum_n |a_\alpha^{(n)} \xi_n| < +\infty \quad (\alpha = 1, 2, ...)$$

with seminorms $\| \cdot \|_\alpha$.

By σ we shall denote the topological direct sum of a countable number of spaces $l(n^\alpha)$, i.e. the space of all sequences

$$y = \{\xi^{(1)}, \xi^{(2)}, ...\}, \quad \text{where } \xi^{(\beta)} = (\xi_n^{(\beta)}) \in l(n^\alpha) \quad (\beta = 1, 2, ...),$$

with seminorms

$$\|y\|_{\alpha,\beta} = \sum_{n=1}^{\infty} |\xi_n^{(\beta)} n^\alpha|. \tag{3}$$

The spaces $\mathcal{E}(R)$, $\mathcal{E}(I)$, $\mathcal{E}(K)$, σ are nuclear. The necessary and sufficient condition for the nuclearity of the Köthe spaces $l(c_\alpha^{(n)})$ is the following ([1], Part 2, p. 59): for every α there exists $\alpha' > \alpha$ such that

$$\sum_{n \in N_{\alpha'}} c_\alpha^{(n)} / c_{\alpha'}^{(n)} < +\infty, \quad \text{where } N_{\alpha'} = \{n : c_{\alpha'}^{(n)} \neq 0\}.$$

This condition is easily derived from I. M. Gelfand' s definition of nuclearity (see [2]).

Remark 1. The spaces $\mathcal{E}(K)$ and $\mathcal{E}(I)$ are isomorphic to the Köthe space $l(n^\alpha)$. The required isomorphic mapping is given, respectively, by the formulas

$$A\xi = \sum_{k=1}^{\infty} \xi_k S_k; \quad B\xi = \sum_{k=1}^{\infty} \xi_k T_k \quad (\xi \in l(n^\alpha)), \tag{4}$$

where S_k are the functions of the trigonometric system, and T_k are the Chebyshev polynomials.

2. Theorem 1. *Every metrizable complete nuclear space with a basis is isomorphically embeddable in the space $\mathcal{E}(R)$.*

Recall that a **basis** (an unconditional basis) of a space X is a sequence of vectors (e_n) from X such that every vector $x \in X$ has a unique representation $x = \sum_n t_n e_n$, where the series $\sum_n t_n e_n$ converges (converges unconditionally) in the topology of the space (see [4], Ch. IV).

Theorem 1 follows directly from the following propositions:

Proposition 1. *The spaces $\mathcal{E}(R)$ and σ are isomorphically embeddable in one another.*

Proof. 1°. The embedding of $\mathcal{E}(R)$ in σ is given by the formula $U_1(x) = \{Ax_0^{-1}, Ax_1^{-1}, A^{-1}x_{-1}, A^{-1}x_2, ...\}$, $x \in \mathcal{E}(R)$, where $x_k \in \mathcal{E}(I)$, $x_k(t) = x(t+2k)$ for $t \in [-1, 1]$ and $k = 0, \pm 1, \pm 2, ...$, A is the operator defined by formula (4).

2°. Let φ_p $(p = 1, 2, ...)$ be fixed functions from $\mathcal{E}(R)$ such that $\varphi_p(t) = 1$ for $|t - 4p\pi| \leq \pi$, $\varphi_p(t) = 0$ for $|t - 4p\pi| \geq 2\pi$. Let

$$U_2 y = \sum_p \widetilde{\varphi}_p B(\xi^{(p)}), \quad y = \{\xi^{(1)}, \xi^{(2)}, ...\} \in \sigma,$$

where B is the operator defined by formula (4), $\widetilde{\varphi}_p : \mathcal{E}(K) \to \mathcal{E}(R)$ is the operator of multiplication of a function from $\mathcal{E}(K)$ by the function φ_p $(p = 1, 2, ...)$. U_2 realizes an isomorphic embedding of the space σ in $\mathcal{E}(R)$. This follows from the fact that the operators $\widetilde{\varphi}_p$ map the space $\mathcal{E}(K)$ isomorphically onto such subspaces of the space $\mathcal{E}(R)$ that the supports of functions belonging to different subspaces do not intersect.

* **Note added in proof.** As B. S. Mityagin showed (in July 1960), these spaces are isomorphic.

Proposition 2. *If the sequence (e_n) is a basis in a metrizable complete nuclear space, then (e_n) is also an unconditional basis.*

This proposition was proved by A. S. Dynin and B. S. Mityagin in ([5]).

Proposition 3 (Rolewicz ([6])). *Let X be a complete metrizable nuclear space whose topology is defined by a system of seminorms*

$$\|x\|_1 \leq \|x\|_2 \leq \cdots,$$

(e_n) an unconditional basis in X. Then X is isomorphic to the Köthe space $l(\|e_n\|_\alpha)$.

Proof. Let $\xi = (\xi_n) \in l(\|e_n\|_\alpha)$. Put $V\xi = \sum_n \xi_n e_n$. Since (e_n) is a basis, the operator V maps $l(\|e_n\|_\alpha)$ one-to-one into X. Let

$$x = \sum_{n=1}^{\infty} \xi_n e_n \tag{5}$$

be an arbitrary vector in X. By assumption, the series (5) converges unconditionally, and, by nuclearity,* we have

$$\sum_n |\xi_n| \, \|e_n\|_\alpha = \sum_n \|\xi_n e_n\|_\alpha < +\infty \qquad (\alpha = 1, 2, \ldots);$$

therefore $(\xi_n) \in l(\|e_n\|_\alpha)$. From Banach' s theorem on the continuity of the inverse operator it follows that V maps $l(\|e_n\|_\alpha)$ isomorphically onto X.

Proposition 4 . *Every nuclear Köthe space is isomorphically embeddable in σ.*

Lemma 1. *Let $l(c_\alpha^{(n)})$ be nuclear. Then there exists a matrix $b_\alpha^{(n)}$ such that: 1) the spaces $l(c_\alpha^{(n)})$ and $l(b_\alpha^{(n)})$ are identically isomorphic, i.e. the classes of sequences forming these spaces coincide; 2) if $\lambda_\alpha^{(n)} = 0$ for $b_{\alpha+1}^{(n)} = 0$, and $\lambda_\alpha^{(n)} = b_\alpha^{(n)}/b_{\alpha+1}^{(n)}$ for $b_{\alpha+1}^{(n)} \neq 0$, then*

$$\sum_n \lambda_\alpha^{(n)} < +\infty \qquad (\alpha = 1, 2, \ldots). \tag{6}$$

The existence of a matrix satisfying conditions 1) and 2) is easily derived from the nuclearity condition for a Köthe space given above.

Lemma 2. *Let the matrix $(b_\alpha^{(n)})$ satisfy conditions 1) and 2) of Lemma 1. Then there exists a matrix $(a_\alpha^{(n)})$ such that the spaces $l(a_\alpha^{(n)})$ and $l(b_\alpha^{(n)})$ are isomorphic and*

$$\text{if } (\xi_n) \in l(a_\alpha^{(n)}), \quad \text{then } (n^\beta \xi_n) \in l(a_\alpha^{(n)}), \qquad \beta = 1, 2, \ldots \tag{7}$$

Proof. From formula (6) it is easy to infer that there exists a sequence (μ_n) of positive numbers such that $\mu_n \geq \lambda_\alpha^{(n)}$ for $n \geq N(\alpha)$, $\alpha = 1, 2, \ldots$; $\sum_n \mu_n < +\infty$. Consider a permutation (p_n) of the indices for which $\mu_{p_1} \geq \mu_{p_2} \geq \cdots$. Put

$$a_\alpha^{(n)} = b_\alpha^{(p_n)} \qquad (\alpha, n = 1, 2, \ldots).$$

Since $\sum_n \mu_{p_n} < +\infty$ and the sequence (μ_{p_n}) is nonincreasing, it follows that

$$d = \sup_n n\mu_{p_n} < +\infty.$$

Let $(\xi_n) \in l(a_\alpha^{(n)})$; since

$$na_\alpha^{(n)} = n\lambda_\alpha^{(p_n)} a_{\alpha+1}^{(n)} \leq da_{\alpha+1}^{(n)}$$

for $p_n \geq N(\alpha)$, we obtain

$$\sum_{n=1}^\infty |n\xi_n a_\alpha^{(n)}| \leq \sum_{p_n < N(\alpha)} |n\xi_n a_\alpha^{(n)}| + d \sum_{p_n \geq N(\alpha)} |\xi_n a_{\alpha+1}^{(n)}| < +\infty.$$

* In complete metrizable nuclear spaces, every unconditionally convergent series converges absolutely ([1]), par

** A similar result is given without proof in ([1]), part 2, p. 68.

Thus, $(n^\beta \xi_n) \in l(a_\alpha^{(n)})$, and then $(n^{2\beta}\xi_n) \in l(a_\alpha^{(n)})$, $(n^{3\beta}\xi_n) \in l(a_\alpha^{(n)}),\cdots$. It is obvious that the mapping $W : W(\xi_n) = (\xi_{p_n})$ is an isomorphism of the space $l(b_\alpha^{(n)})$ onto $l(a_\alpha^{(n)})$.

Proof of Proposition 4. In view of the preceding lemmas it suffices to prove that the space $l(a_\alpha^{(n)})$, satisfying condition (7), is embeddable in σ. Let, for $\xi = (\xi_n) \in l(a_\alpha^{(n)})$, $U\xi = \{(\xi_1 a_1^{(1)}, \xi_2 a_1^{(2)}, \ldots), (\xi_1 a_2^{(1)}, \xi_2 a_2^{(2)}, \ldots), \ldots\}$. It is obvious that the mapping U is additive and homogeneous. From (3) and (7) it follows that $\|\xi\|_\alpha \le \sum_n |\xi_n a_\alpha^{(n)}| \le \sum_n |n^\beta \xi_n a_\alpha^{(n)}| = \|U\xi\|_{\beta,\alpha}$; therefore $U\xi \in \sigma$ and U^{-1} is continuous. Continuity follows from the Banach-Steinhaus theorem, since

$$U\xi = \lim_{k\to\infty} U_k\xi, \qquad U_k\xi = \{(\eta_n^{(1)}(k)), (\eta_n^{(2)}(k)), \ldots\},$$

where $\eta_n^{(\alpha)}(k) = \xi_n a_\alpha^{(n)}$ for $n \le k,\ \alpha \le k$, and $\eta_n^{(\alpha)}(k) = 0$ for the remaining n, α.

3. **Definition 3.** A nuclear space will be called **supernuclear** if, instead of condition (2) of Definition 2, the following conditions are fulfilled for all x:

$$\sum_{n=1}^{\infty} n^\beta \sup\{f_n^{(\alpha)}(x) : \|x\|_\alpha \le 1\} < \infty \qquad (\beta = 1, 2, \ldots); \tag{2a}$$

$$\|x_n^{(\alpha)}\|_\alpha = 1 \qquad (n = 1, 2, \ldots).$$

Various spaces of analytic functions may serve as examples of supernuclear spaces. The spaces $\mathcal{E}(R)$, $\mathcal{E}(I)$, $\mathcal{E}(K)$, σ are not supernuclear.

Theorem 2. *If X is a supernuclear metrizable complete space, then X is isomorphically embeddable in $\mathcal{E}(R)$ and in σ.*

Proof. The required embedding in σ is realized by the mapping $U : X \to \sigma$, given by the formula

$$Ux = \{(f_n^{(1)}(x)), (f_n^{(2)}(x)), \ldots\},$$

where $f_n^{(\alpha)}$ are the linear functionals occurring in Definition 3 and corresponding to the seminorms $\| \ \|_\alpha$ $(n = 1, 2, \ldots)$.

Theorem 3. *Let F be a bounded set in the metrizable nuclear space X, and let $L(F)$ be the linear span of the set F. Then $L(F)$ is isomorphically embeddable in σ and in $\mathcal{E}(R)$.*

Theorem 3 is easily obtained from the following lemma.

Lemma 3. *For each seminorm $\| \ \|_\alpha$ one can find vectors $(y_n^{(\alpha)})$ from X and linear functionals $g_n^{(\alpha)}$ on X such that*

$$\text{for every } x \in L(F) \qquad \lim \left\| x - \sum_{i=1}^{\infty} g_i^{(\alpha)}(x) y_i^{(\alpha)} \right\|_\alpha = 0; \tag{1 }$$

$$\sum n^\beta \sup\{g_n^{(\alpha)}(x) : x \in F\} < +\infty \qquad (\beta = 1, 2, ...); \qquad (2\ )$$
$$\|y_n^{(\alpha)}\|_\alpha = 1 \qquad (n = 1, 2, ...).$$

The proof of Lemma 3 is obtained by means of arguments analogous to the proof of Lemma 7 in the work [7].

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Note: Figure translations are in progress. See original paper for figures.