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Abstract

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ON THE EXISTENCE OF A TRANSFORMATION OPERATOR FOR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

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In the theory of linear differential operators of the second order, an important role is played by the transformation operators

$$y(x,\lambda) = \omega(x,\lambda) + \int_0^x K(x,t)\omega(t,\lambda)\,dt, \tag{1}$$

which transform a solution $\omega(s,\lambda)$ of the equation

$$-\omega'' = \lambda^2\omega$$

into a solution $y(x,\lambda)$ of the equation

$$-y'' + qy = \lambda^2 y.$$

Formula (1) was apparently first obtained by J. Delsarte and then independently of him by A. Ya. Povzner (^{1}). These operators were substantially used by V. A. Marchenko, I. M. Gel' fand, B. M. Levitan, and other authors in works on the spectral theory of differential operators.

Recently L. A. Sakhnovich (^{2}) generalized formula (1) to differential operators of arbitrary order. He showed that if $y(x,\lambda)$ is a solution of the equation

$$y^{(n)} + q(x)y = \lambda^n y$$

under the initial conditions

$$y(0,\lambda) = 1, \quad y^k(0,\lambda) = 0 \quad (k = 1, \dots, n-1),$$

and $\omega(x,\lambda)$ is a solution of the equation

$$\omega^{(n)} = \lambda^n \omega$$

under the same initial conditions, then formula (1) is valid on some interval of variation of the variable x $(0 \le x \le l)$, if $q(z)$ is a function analytic in the disk $|z| \le R$, whose radius depends on l and on the order n of the equation. Moreover, if $q(z)$ is an entire function, then the kernel $K(x,t)$ is also an entire function of x and t.

Here we show that the assumption of analyticity of the function $q(x)$ is essential. Namely, the following holds.

Theorem. *If*

$$q(x) = \begin{cases} q_0(x), & 0 \le x \le \delta, \\ 0, & x > \delta, \end{cases}$$

where $q_0(x) \not\equiv 0$ is an entire function, and δ is some positive number, then the representation (1) is impossible for values of x exceeding some N_δ, depending on the order of the equation and on the number δ.

Proof. To simplify the notation, we put $n = 4$. We have

$$y(x,\lambda) = \omega(x,\lambda) - \int_0^x \frac{S(\lambda(x-t))}{\lambda^3} q(t) y(t,\lambda)\, dt, \tag{2}$$

where

$$S(u) = \frac{e^u - e^{-u} + ie^{iu} - ie^{-iu}}{4}.$$

It is known (see, for example, (^{3})) that the solution $y(x,\lambda)$ is an entire function of finite degree in λ, whose indicator diagram is the square Q_x with vertices at the points $(\pm x, \pm ix)$. By a well-known theorem of D. Pólya, its Laplace transform $\varphi_x(z)$ is a holomorphic function outside this square, and moreover $\varphi_x(\infty) = 0$.

Suppose that $x > 2\delta$, and let us show that in this case the Laplace transform of the integral on the right-hand side of (2) can be represented in the form

$$\theta(\zeta) = \int_0^\delta q_0(t) J_{x-t} \varphi_t(\zeta)\, dt, \tag{3}$$

where

$$J_a\varphi_t(\zeta) = \frac{1}{8}\sum_{k=0}^{3} i^k \int_0^{i^{ka}} (i^{ka} - z)\varphi_t(\zeta + z)\,dz. \tag{4}$$

Indeed, the function $\theta(\zeta)$ defined by equality (3) is holomorphic outside the circle of radius x and vanishes at infinity. Such a function is the Laplace transform of the function

$$\psi(\lambda) = \frac{1}{2\pi i}\int_C \theta(\zeta)e^{\lambda\zeta}\,d\zeta =$$

$$= \frac{1}{2\pi i}\int_C e^{\lambda\zeta}\,d\zeta \int_0^{\delta} q_0(t)\,dt \cdot \frac{1}{8}\sum_{k=0}^{3} i^k \int_0^{i^k(x-t)} (i^k(x-t) - z)^2\varphi_t(\zeta + z)\,dz.$$

Changing the order of integration, we obtain

$$\psi(\lambda) = \int_0^{\delta} q_0(t)\,dt \cdot \frac{1}{8}\sum_{k=0}^{3} i^k \int_0^{i^k(x-t)} (i^k(x-t) - z)^2\,dz \cdot \frac{1}{2\pi i}\int_C e^{\lambda\zeta}\varphi_t(\zeta + z)\,d\zeta.$$

Observing further that

$$y(t,\lambda) = \frac{1}{2\pi i}\int \varphi_t(\zeta)e^{\lambda\zeta}\,d\zeta,$$

we transform the function $\psi(\lambda)$ into the form

$$\psi(\lambda) = \int_0^{\delta} q_0(t)\,dt \cdot \frac{1}{8}\sum_{k=0}^{3} i^k \int_0^{i^k(x-t)} (i^k(x-t) - z)^2 e^{-\lambda z}\,dz \cdot y(t,\lambda).$$

Computing the inner integral, we finally obtain

$$\psi(\lambda) = \int_0^{\delta} q_0(t)\,dt\,\frac{S(\lambda(x-t))}{\lambda^3}\,y(t,\lambda) = \int_0^{x} \frac{S(\lambda(x-t))}{\lambda^3}\,y(t,\lambda)q(t)\,dt.$$

Taking the Laplace transform of both sides of (2), we obtain the equality

$$\varphi_x(\zeta) = \sum_{0}^{3} \frac{1}{\zeta - i^{kx}} - \int_0^{\delta} (J_{x-t}\varphi_t(\zeta))q_0(t)\,dt, \tag{5}$$

which holds everywhere outside the square (Q_x).

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

Since $q_0(x)$ is an entire function, by L. A. Sakhnovich' s theorem, for $x < \delta$ formula (1) is valid. Taking the Laplace transform of both sides of equality (1) and replacing x by t, we shall have, for $t < \delta$,

$$\varphi_t(\zeta) = \sum_0^3 \left(\frac{1}{\zeta - i^k t} + \int_0^t \frac{k(t,s)}{\zeta - i^k s}\, ds \right). \tag{6}$$

It follows from this representation that the function $\varphi_t(\zeta)$ is holomorphic in the $\zeta = \xi + i\eta$ plane cut along the cross Kt, consisting of the segments
$-t \le \xi \le t$ $(\eta = 0)$, $-t \le \eta \le t$ $(\xi = 0)$.

Let us find the domain of holomorphy of the function $J_a\varphi(\zeta)$. To this end we write the first term $P_1\varphi(\zeta)$ of the right-hand side of (4) in the form

$$P_1\varphi(\zeta) = \frac{1}{8} \int_\zeta^{\zeta+a} (a + \zeta - \tau)^2 \varphi(\tau)\, d\tau.$$

Fig. 1

It is obvious that if the straight-line segment $(\zeta, \zeta + a)$ does not intersect the cross K_t, then the function $P_1\varphi(\zeta)$ is holomorphic. Thus $P_1\varphi(\zeta)$ is holomorphic everywhere outside the cross K_t of the rectangle $(|\eta| \le t,\ -a \le \xi \le 0)$ and the segment FE $(-a - t \le \xi \le -t)$ (see Fig. 1). Through the sides DC and AB of the rectangle this function can be continued up to the real axis. Indeed, this continuation can be carried out by deforming the contour of integration as shown in Fig. 1. Thus $P_1\varphi$ is holomorphic in the ζ-plane cut along the segments FG, AD, and BC.

Fig. 2

The domains of holomorphy of the other terms in (4) are obtained analogously. Finally we obtain that the function $J_{x-t}\varphi_t(\zeta)$ $(t < \delta)$ is holomorphic in the plane with cuts shown in Fig. 2.

Let us now compute the difference of the limiting values of the function $J_{x-t}\varphi_t(\zeta)$ on opposite sides of the cut AC. For this, note that it suffices to find the difference of the limiting values of only the first term $P_1\varphi$, since the remaining ones are holomorphic on this cut. The limiting values on the left side of the cut $J_+(z)$ are obtained by integration over the segment $[z, z + (x - t)]$, while the

Fig. 3

Figure 3: Fig. 3

limiting values on the right side of the cut $J_{\_}(z)$ are obtained by integration over the deformed contour. Therefore the difference of the limiting values $\Phi(z) = J_+(z) - J_{\_}(z)$ is equal to the integral of the function $\varphi_t(\zeta)(a+z-\zeta)^2$ over the closed contour L, enclosing a part of the segment BC (in Fig. 1). From the representation (6) of the function $\varphi_t(\zeta)$ it follows that this integral exists and is equal to

$$\Phi(z) = \left(z + (x-t) - it\right)^2 + \int_{\frac{z+(x-t)}{i}}^{t} K(t,s)\left(z + (x-t) - is\right)^2 ds, \tag{7}$$

where z lies on the segment AC (Fig. 2).

Since, by L. A. Sakhnovich' s theorem, the kernel $K(t,s)$ is an entire function of both variables, it is clear from (7) that $\Phi(z)$ is also an entire function. Thus, $J_+(z) - \Phi(z)$ and $J_{\_}(z)$ are the limiting values of analytic functions defined on different sides of the cut. These

the limiting values are equal. Therefore the function $J_{\_}(z)$ is continued across the cut AC up to the cut CB (see Fig. 2). It is obvious that everywhere and, in particular, on CB, the equality $J_+(z) - J_{\_}(z) = \Phi(z)$ holds. Hence it follows that the function

$$\theta(\zeta) = \int_0^{\delta} (J_{x-t}\varphi_t(\zeta)) q_0(t)\, dt$$

is holomorphic in the domain G, which is obtained by making in the z-plane the cuts shown in Fig. 3, and the difference of the values of this function on the sides of the cut BC' is equal to

$$\int_{\frac{z+x}{1+i}}^{\delta} q_0(t)\, dt \left[(z + (x-t) - it)^2\right] + \int_{\frac{z+(x-t)}{i}}^{t} K(t,s)(z + (x-t) - s)^2\, ds.$$

Fig. 3

We note that, for $x > 2\delta$, a passage remains between the cuts BC' and $C''B'$, i.e. the cuts do not separate the plane. But if formula (1) holds for some $x > 2\delta$, then the equality

$$\varphi_x(\zeta) = \sum_{k=0}^{3} \left(\frac{1}{\zeta - i^k x} + \int_0^x \frac{K(x,s)}{\zeta - i^k s}\, ds \right), \tag{4'}$$

will hold, from which follows the holomorphy of the function $\varphi_x(z)$ everywhere outside the cross K_x.

We shall now use equality (5). It holds by virtue of the analyticity of its right- and left-hand sides everywhere in the domain G. The difference of the limiting values of the function $\varphi_x(\zeta)$ on the sides of the cut BC is equal to zero. By virtue of (5) this is also true for $\theta(\zeta)$. Therefore

$$\int_{\frac{z+x}{1+i}}^{\delta} q_0(t)\,dt\,(z+(x-t)-it)^2 + \int_{\frac{z+(x-t)}{i}}^{t} K(t,s)(z+(x-t)-is)^2\,ds = 0.$$

Putting

$$\frac{z+x}{1+i} = w,$$

we rewrite this relation in the form

$$\int_{w}^{\delta} q_0(t)\,dt\left[(w-t)^2 + \int_{\frac{(1+i)w-t}{i}}^{t} K(t,s)\left(w-\frac{t+is}{1+i}\right)^2 ds\right] = 0.$$

Differentiating both sides of it 3 times with respect to w, we obtain

$$-q_0(w) - (1-i)\int_{w}^{\delta} q_0(t)K\left(t, \frac{(1+i)w-t}{i}\right) dt = 0.$$

Thus $q_0(w)$ satisfies a homogeneous Volterra equation and, consequently, is equal to zero, which contradicts the conditions of the theorem. Thus the representation (1) is impossible. The theorem is proved.

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