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Abstract

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ON SOME PROPERTIES OF THE SPECTRUM OF A NON-SELF-ADJOINT SECOND-ORDER DIFFERENTIAL OPERATOR

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Consider the equation

$$y'' + p(x)y' + [q(x) - \lambda]y = 0 \tag{1}$$

with continuous complex periodic coefficients, $p(x+1) = p(x)$, $q(x+1) = q(x)$. By the spectrum of equation (1) we shall mean the set of complex values of the parameter λ for which equation (1) has at least one solution bounded along the entire real axis.

It is known ([1]) that for every λ equation (1) has a solution $y(x)$ satisfying, for all x, the condition

$$y(x+1) = sy(x),$$

where s is any one of the roots of the characteristic equation

$$s^2 - F(\lambda)s + \exp\left[-\int_0^1 p(x)\,dx\right] = 0. \tag{2}$$

Here $F(\lambda) = y_1(1,\lambda) + y_2'(1,\lambda)$; $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are a fundamental system of solutions of equation (1), satisfying the initial conditions

$$y_1(0,\lambda) = 1, \quad y_2(0,\lambda) = 0,$$
$$y_1'(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1.$$

The function $F(\lambda)$ is an entire analytic function of the parameter λ. In order that the point λ_0 belong to the spectrum, it is necessary and sufficient that the modulus of at least one of the roots s_1, s_2 of equation (2) be equal to one.

We shall call a point λ a point of type + if, for this value of λ, the moduli of the roots s_1, s_2 of equation (2) are both greater than or both less than one, and a point of type $-$ if, for this value of λ, the modulus of one of the roots s_1, s_2 is greater than one and the modulus of the other is less than one.

The following assertions can be proved.

1. If $\int_0^1 \operatorname{Re} p(x)\,dx = 0$, then the set of points of type $+$ is empty; if

$$\int_0^1 \operatorname{Re} p(x)\,dx \neq 0,$$

then there exist both points of type $-$ and points of type $+$. Both the set of points of type $+$ and the set of points of type $-$ are open sets, each connected component of which (we shall call it a domain of type $+$, or, respectively, a domain of type $-$) has as its boundary a continuum of spectral points.

2. If $\int_0^1 \operatorname{Re} p(x)\,dx \neq 0$, then in any neighborhood of each point of the spectrum there are both points of type $+$ and points of type $-$. It follows that if two domains have a segment of the spectrum as a common boundary, then these domains are of different types.

3. Every domain of type $-$ is unbounded; hence every bounded domain of type $+$ is simply connected.

The spectral problem considered here is equivalent to the problem of the aggregate of the spectra of the operators defined on the interval $[0, 1]$ by equation (1) and the boundary conditions

$$y(1) = e^{it} y(0); \qquad y'(1) = e^{it} y'(0) \tag{3}$$

for all real t. For each fixed t, this operator has a countable set of eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$, whose disposition in the λ-plane obeys the known asymptotics (2). As t varies in the conditions (3), the image of the circle $s = e^{it}$ will be the curves $\lambda_n(s)$, whose union forms the spectrum of equation (1). There are two essentially different cases.

A. If $\int_0^1 \operatorname{Re} p(x)\,dx = 0$, then the spectrum consists of a finite or countable set of components, each of which is homeomorphic to an interval and is its analytic image. An example is provided by the well-studied self-adjoint case $p(x) \equiv \operatorname{Im} q(x) \equiv 0$, when the spectrum consists of intervals of the real axis.

B. If $\int_0^1 \operatorname{Re} p(x)\,dx \neq 0$, then the spectrum consists of a finite number of components, each of which, in the extended λ-plane, is homeomorphic to a circle and is its analytic image. A trivial example is the case of constant p and q, where $\operatorname{Re} p \neq 0$. In this case the spectrum is a parabola in the λ-plane. With regard to the mutual position of these components, assertions 1–3 allow one to draw the following conclusions. Two components of the spectrum may, generally speaking, have a common point, which will be a multiple point of the spectrum.

But if at least one of the two components is bounded, then two such components cannot have more than one common point, since this would lead to the formation of two adjacent bounded domains, which is excluded by assertions 1–3. Nor can either of two bounded components lie inside the other, for the same reason. Several components may, generally speaking, form a chain, where each following component has a common point with the preceding one. But from the same considerations it is evident that such a chain cannot be "closed" in the finite part of the λ-plane.

Some metric characteristics of the spectrum can be obtained from the following considerations.

The substitution $u(x) = \exp\left[\int_0^x f(x)\,dx\right] y(x)$, where $f(x)$ is a continuous complex function with period 1, transforms equation (1) into the form

$$u'' + [p(x) - 2f(x)]u' + [q(x) + f^2(x) - p(x)f(x) - f'(x) - \lambda]u = 0. \quad (4)$$

Let the point λ_0 be a point of the spectrum of equation (1). Consider the number

$$A = \int_0^1 \operatorname{Re} f(x)\,dx \int_0^1 [\operatorname{Re} f(x) - \operatorname{Re} p(x)]\,dx.$$

It is easy to verify the validity of the assertions:

a) If $A = 0$, then λ_0 is also a point of the spectrum for equation (4). The spectra of equations (1) and (4) coincide.

b) If $A > 0$, then λ_0 is a point of type + for equation (4). In this case, each component of the spectrum of equation (1) lies inside some component of the spectrum of equation (4), and inside each component of the spectrum of equation (4) there lies at least one component of the spectrum of equation (1). We shall say that in this case the spectrum of equation (4) encloses the spectrum of equation (1).

c) If $A < 0$, then λ_0 is a point of type − for equation (4). In this case the spectrum of equation (1) encloses the spectrum of equation (4).

If $f(x) = p(x)/2$, then we arrive at the equation

$$u'' + \left[q(x) - \frac{p^2(x)}{4} - \frac{p'(x)}{2} - \lambda\right] u = 0. \quad (5)$$

The spectrum of this equation, as follows from ([3]), lies in the strip

$$\operatorname{Inf}\operatorname{Im}\left[q(x) - \frac{p^2(x)}{4} - \frac{p'(x)}{2}\right] \leqslant \operatorname{Im}\lambda \leqslant \operatorname{Sup}\operatorname{Im}\left[q(x) - \frac{p^2(x)}{4} - \frac{p'(x)}{2}\right]. \quad (6)$$

In this case $A < 0$, and, consequently, the spectrum of equation (1) encloses the spectrum of equation (5), whence it follows that each component of the spectrum of equation (1) has a nonempty intersection with the strip (6).

Let us note that the theorem on the number of zeros of a solution proved in ([3]) is also valid for equation (1), namely: no solution $y(x)$ of equation (1) can vanish at more than $2n+3$ points of the real interval (a, b), if n is the sum of the number of minima and the number of maxima of the function

$$\operatorname{Im}\left[q(x) - \frac{p^2(x)}{4} - \frac{p'(x)}{2}\right]$$

in this interval, and function (7) is not equal to a constant on any subinterval of (a, b).

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CITED LITERATURE

[1] E. Kamke, *Handbook of Ordinary Differential Equations*, IL, 1951.
[2] M. A. Naimark, *Linear Differential Operators*, 1954.
[3] M. I. Serov, *Uspekhi Mat. Nauk*, 6, 6 (1951).

Note: Figure translations are in progress. See original paper for figures.