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Abstract

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ON THE CENTRAL LIMIT THEOREM FOR ERGODIC ENDOMORPHISMS OF COMPACT COMMUTATIVE GROUPS

(Presented by Academician A. N. Kolmogorov, 22 VI 1960)

In papers ([1–3]) it was proved by different methods that if $g(x)$ is a real periodic function with period one with Fourier coefficients

$$C_n(g) = \int_0^1 g(x)e^{-2\pi inx}\,dx = O\left(\frac{1}{|n|^\beta}\right), \qquad \beta > \frac{1}{2},$$

or satisfying a Hölder condition with exponent $\alpha > 0$,

$$m_g = \int_0^1 g(x)\,dx,$$

a is an arbitrary integer not equal to ± 1, and μ is Lebesgue measure on the interval $[0, 1]$, then*

$$\mu\left\{x : \frac{1}{\sqrt{p}}\left(\sum_{t=0}^{p-1} g(a^t x) - m_g p\right) < y\right\} \xrightarrow[p\to\infty]{} \frac{1}{\sqrt{2\pi}\sigma_g}\int_{-\infty}^{y} e^{-z^2/2\sigma_g^2}\,dz$$

under the additional condition

$$\sigma_g^2 = \int_0^1 (g(x) - m_g)^2\,dx + 2\sum_{t=1}^{\infty}\int_0^1 (g(a^t x) - m_g)(g(x) - m_g)\,dx > 0.$$

The present paper generalizes the indicated results. It uses the basic definitions and notation of ([5]). Let G be a compact commutative group with invariant measure μ, and let T be an endomorphism of the group G into G. For $g \in L^2(G)$ denote

$$m_g = \int_G g(x)\,d\mu$$

and consider the question of when the quantity

$$\frac{1}{\sqrt{p}}\left(\sum_{t=0}^{p-1} g(T^t x) - m_g p\right) = \frac{1}{\sqrt{p}}\left(\sum_{t=0}^{p-1} U^t g(x) - m_g p\right)$$

as $p \to \infty$ is asymptotically normal or degenerate, i.e., when for some $\sigma \geq 0$ the limiting relation holds

$$\lim_{p\to\infty} \mu\left\{x : \frac{1}{\sqrt{p}}\left(\sum_{t=0}^{p-1} g(T^t x) - m_g p\right) < y\right\} = \Phi_\sigma(y) \tag{1}$$

* In ([4]) the same result was proved under weaker restrictions on the function $g(x)$.

for all points of continuity y of the function $\Phi_\sigma(y)$, where

$$\Phi_\sigma(y) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{y} e^{-z^2/2\sigma^2}\,dz, \quad \text{if } \sigma > 0;$$
$$\Phi_0(y) = \begin{cases} 0, & y \leqslant 0, \\ 1, & y > 0. \end{cases} \tag{2}$$

If for a function $g(x)$ relation (1) holds with $\sigma \geqslant 0$, then we shall say that $g(x)$ satisfies the central limit theorem (c.l.t.).

By $\chi(x)$ we shall denote the characters of the group G. As is known (see, for example, ([6])), functions $g \in L^2(G)$ are expanded in a Fourier series in characters, convergent in mean square:

$$g(x) = \sum_n C_n(g)\chi_n(x);$$

in what follows it is assumed everywhere that $\chi_0(x) = 1$.

For convenience of notation we introduce an operator A^*, mapping the set of indices of characters $\mathfrak{N}$ into itself and defined by the formula

$$\chi_{A^* n} = U\chi_n,$$

so that $\chi_{A^{*t} n} = U^t \chi_n$ for all nonnegative integers t.

Theorem 1. Let G be a compact commutative group with invariant measure μ, let T be its ergodic endomorphism, $g(x) \in L^2(G)$, real-valued, and

$$\sum_{t=0}^{\infty}\sum_{n\neq 0} |C_n(g)|\;|C_{A^{*t} n}| < \infty; \tag{3}$$

then $g(x)$ satisfies the c.l.t.; moreover,

$$\sigma^2 = \sigma_g^2 = \|g - m_g\|^2 + 2\sum_{t=1}^{\infty}((U^t g - m_g), (g - m_g)) = \tag{4}$$

$$= \sum_{n\neq 0} |C_n(g)|^2 + 2\sum_{t=1}^{\infty}\sum_{n\neq 0} C_n(g)\overline{C}_{A^{*t}n}(g). \tag{4'}$$

Lemma 1. Let

$$b_p(g) = \int_G \left(\sum_{t=0}^{p-1} U^t g(x) - m_g p\right)^2 d\mu.$$

If (3) is fulfilled, then

$$\lim_{p\to\infty} \frac{b_p(g)}{p} = \sigma_g^2.$$

Proof of the theorem. For $\sigma_g = 0$ the theorem is trivial. Let $\sigma_g > 0$. Take G to be the space of elementary events and μ to be the probability distribution on it. Then the function

$$\xi(t, x) = U^t g(x) = g(T^t x)$$

forms a real stationary random process with discrete time $t = 0, 1, 2, \ldots$. We first consider the case when

$$g(x) = \sum_{n\in\mathfrak{N}'} C_n(g)\chi_n(x),$$

where $\mathfrak{N}' \subset \mathfrak{N}$ is finite. Then $g'(x)$ is bounded, and all moments of $\xi(t, x)$ exist:

$$m_\xi^{(k)}(t_1, \ldots, t_k) = \int_G U^{t_1} g'(x) \cdots U^{t_k} g'(x)\, d\mu =$$
$$= \sum_{\substack{n_1,\ldots,n_k\in\mathfrak{N}' \\ U^{t_1}\chi_{n_1}\cdots U^{t_k}\chi_{n_k}=1}} C_{n_1}(g)\cdots C_{n_k}(g). \tag{5}$$

Since $\mathfrak{N}'$ is finite, it follows from (5) that, for fixed k, the set of values $\mathfrak{M}_k$ of the function $m_\xi^{(k)}(t_1, \ldots, t_k)$ consists of a finite number of points. But the semi-invariants $s_\xi^{(k)}(t_1, \ldots, t_k)$ of the process $\xi(t)$ (see (7), § 1) are expressed in terms of the moments $m_\xi^{(l)}(t_{i_1}, \dots, t_{i_l})$, $1 \le l \le k$, $1 \le i_s \le k$, by formula II-c from ([8]), which contains only a finite sum of products of a finite number of moments. Therefore, for fixed k, the set of values $\mathfrak{S}_k$ of the function $s_\xi^{(k)}(t_1, \dots, t_k)$ also consists only of a finite number of points.

But, by the theorem of V. A. Rokhlin (([5]), § 3), every ergodic endomorphism of a compact commutative group is mixing of all degrees; hence, according to Theorem 6 in ([9]), it follows that

$$s_\xi^{(k)}(t_1, \dots, t_k) \underset{\max_{i,j}|t_i - t_j| \to \infty}{\longrightarrow} 0$$

for every $k \ge 2$. Since $\mathfrak{S}_k$ is finite, it follows from this that for every $k \ge 2$ there exists $D_k < \infty$ such that

$$s_\xi^{(k)}(t_1, \dots, t_k) = 0$$

when $\max_{i,j}|t_i - t_j| > D_k$. Consequently,

$$\sum_{t_1, \dots, t_k = 0}^{p-1} s_\xi^{(k)}(t_1, \dots, t_k) = O(p) = o(bp^{k/2})$$

for $k \ge 3$, since, by Lemma 1, $b_p = \sigma_g^2 p + o(p)$ and $\sigma_g > 0$. Hence, by Theorem 7 in ([9]), relation (1) with $\sigma = \sigma_g$ follows, and for the case in which only a finite number of Fourier coefficients $C_n(g)$ are different from zero, the theorem is proved.

The theorem is extended without difficulty to the general case by using Lemma 1.

Corollary 1. If

$$\sum_n |C_n(g)| < \infty,$$

then $g(x)$ satisfies the central limit theorem for every ergodic endomorphism.

Let us now consider the special case where G is the k-dimensional torus: $x = (x_1, \dots, x_k)$, $0 \le x_i < 1$ (see ([5]), example 1, § 4). Let

$$\omega_{x_i}^{(2)}(\delta, g) =$$

$$= \sup_{0 \le h \le \delta} \left(\int_0^1 \cdots \int_0^1 \left| g(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_k) - g(x_1, \dots, x_k) \right|^2 dx_1 \cdots dx_k \right)^{1/2}.$$

Theorem 1 allows us to obtain the following theorem:

Theorem 2. Let G be the k-dimensional torus; μ the Lebesgue measure on it; A an integer square nonsingular matrix of order k, among whose characteristic roots there are no roots of unity; $g(x)$ a real-

a real function from $L^2(G)$, periodic with period one in each variable x_i, $1 \leq i \leq k$, and such that for all $1 \leq i \leq k$

$$\omega_{x_i}^{(2)}(\delta, g) \leq \frac{C}{(\ln \frac{1}{\delta})^{\alpha}}, \qquad \alpha > 1.$$

Then

$$\sum_{t=1}^{\infty} \left|(g(A^t x) - m_g), (g(x) - m_g)\right| < \infty; \tag{6}$$

$$\lim_{p\to\infty} \mu \left\{ x : \frac{1}{\sqrt{p}} \left(\sum_{t=0}^{p-1} g(A^t x) - m_g p \right) < y \right\} \Phi_{\sigma_g}(y) \tag{7}$$

for all points of continuity of the function $\Phi_{\sigma_g}(y)$, where $\Phi_{\sigma}(y)$ is defined by formula (2), and σ_g by formula (4).

Remark 1. In terms of the rate of decrease of the Fourier coefficients of the function $g(x)$

$$C_{n_1 \dots n_k}(g) = \int_0^1 \cdots \int_0^1 g(x) e^{-2\pi i (n,x)}\, dx_1 \cdots dx_k$$

a sufficient condition for $g(x)$ to satisfy the central limit theorem can be expressed as follows: if the matrix A satisfies the conditions of Theorem 2 and

$$|C_{n_1 \dots n_k}(g)| \leq B \prod_{i=1}^{k} \frac{1}{(1 + |n_i|)^{1/2} (\ln(2 + |n_i|))^{\beta}},$$

where $B < \infty$, $\beta > 3/2$, then relations (6) and (7) hold.

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Note: Figure translations are in progress. See original paper for figures.