# V. L. FAINSHMIDT

1960

SovietRxiv

## Abstract

## Full Text

# V. L. FAINSHMIDT

# ON A CLASS OF REGULARLY MONOTONE POLYNOMIALS

*(Presented by Academician S. N. Bernstein on 22 X 1959)*

Following S. N. Bernstein ($^1$), we shall call a function $f(x)$ **regularly monotone of order** $m$ on $[0,1]$ if it and its first $m$ derivatives do not change sign on the interval $[0,1]$. Every regularly monotone function is characterized by a sequence of type numbers $\lambda_1, \lambda_2, \dots, \lambda_s$, which are defined as follows. For definiteness, we shall assume that $f(x)f'(x) \geq 0$ for $x \in [0,1]$. Then $\lambda_1$ has the property that $f^{(i-1)}(x)f^{(i)}(x) \geq 0$ for $i \leq \lambda_1$ and $f^{(\lambda_1)}(x)f^{(\lambda_1+1)}(x) \leq 0$. The type number $\lambda_2$ is determined from the condition $f^{(i-1)}(x)f^{(i)}(x) \leq 0$ for $\lambda_1 < i \leq \lambda_1+\lambda_2$ and $f^{(\lambda_1+\lambda_2)}(x)f^{(\lambda_1+\lambda_2+1)}(x) \geq 0$. The numbers $\lambda_3, \lambda_4, \dots, \lambda_s$ are defined analogously, and one must have $\lambda_1 + \lambda_2 + \dots + \lambda_s = m$.

Consider the class ${}_{2,m}$ of polynomials regularly monotone of order $m$ on $[0,1]$, for which the first and last type numbers are equal to 1 or 2, while all the remaining type numbers are equal to 2. If the polynomial $P_n(x) \in {}_{2,m}$, then, obviously, for it and its first two derivatives the following possibilities are admissible:

$$\begin{array}{lll} 1) & P_n(x)P_n'(x) \geq 0, & P_n'(x)P_n''(x) \leq 0; \\ 2) & P_n(x)P_n'(x) \leq 0, & P_n'(x)P_n''(x) \leq 0; \\ 3) & P_n(x)P_n'(x) \leq 0, & P_n'(x)P_n''(x) \geq 0; \\ 4) & P_n(x)P_n'(x) \geq 0, & P_n'(x)P_n''(x) \geq 0. \end{array} \tag{A}$$

We divide the class ${}_{2,m}$ into four subclasses ${}^{(i)}_{2,m}$ $(i = 1,2,3,4)$; moreover, we shall say that $P_n(x) \in {}^{(i)}_{2,m}$ if $P_n(x) \in {}_{2,m}$ and satisfies the $i$-th of conditions (A). It is clear that

$${}_{2,m} = {}^{(1)}_{2,m} + {}^{(2)}_{2,m} + {}^{(3)}_{2,m} + {}^{(4)}_{2,m}.$$

The most interesting in the class ${}_{2,m}$ are the polynomials $A_{i,m}(x) \in {}^{(1)}_{2,m}$ $(i = 1,2,3,4)$, satisfying the conditions

$$A_{1,m}^{(4k)}(0) = A_{1,m}^{(4k+1)}(1) = A_{1,m}^{(4k+2)}(1) = A_{1,m}^{(4k+3)}(0) = 0;$$

$$A_{2,m}^{(4k)}(1) = A_{2,m}^{(4k+1)}(1) = A_{2,m}^{(4k+2)}(0) = A_{2,m}^{(4k+3)}(0) = 0;$$

$$A_{3,m}^{(4k)}(1) = A_{3,m}^{(4k+1)}(0) = A_{3,m}^{(4k+2)}(0) = A_{3,m}^{(4k+3)}(1) = 0;$$

$$A_{4,m}^{(4k)}(0) = A_{4,m}^{(4k+1)}(0) = A_{4,m}^{(4k+2)}(1) = A_{4,m}^{(4k+3)}(1) = 0$$

and normalized by the condition

$$A_{i,m}^{(m)}(x) = 1.$$

These polynomials, as is not difficult to show, are connected with one another by the relations

$$A_{i,m}^{(4k+j)}(x) = A_{i+j,m-4k-j}(x),$$

$$A_{i,m}(x) = (-1)^m A_{i+2,m}(1-x),$$

where in the last formulas $A_{k,p}(x) \equiv A_{k-4,p}(x)$ if $k > 4$.

The polynomials $A_{i,m}(x)$ are constructed by the method indicated by S. N. Bernstein in the paper ($^2$). For the construction, we introduce the numbers $E_m^{(i)}$ $(i = 1, 2, 3, 4)$, analogous to the Euler-Bernstein numbers and defined by the equalities

$$E_0^{(i)} = 1 \qquad (i = 1, 2, 3, 4);$$

$$E_{4k}^{(1)} = E_{4k+1}^{(1)} = 0; \quad (1 + E^{(1)})_{4k+2} = (1 + E^{(1)})_{4k+3} = 0;$$

$$E_{4k+1}^{(2)} = E_{4k+2}^{(2)} = 0; \quad (1 + E^{(2)})_{4k+3} = (1 + E^{(2)})_{4k} = 0;$$

$$E_{4k+2}^{(3)} = E_{4k+3}^{(3)} = 0; \quad (1 + E^{(3)})_{4k} = (1 + E^{(3)})_{4k+1} = 0;$$

$$E_{4k+3}^{(4)} = E_{4k}^{(4)} = 0; \quad (1 + E^{(4)})_{4k+1} = (1 + E^{(4)})_{4k+2} = 0,$$

in which the expression $(1 + E)_m$ means that the brackets are to be expanded according to Newton’ s binomial formula and the powers $E^r$ replaced by the numbers $E_r$.

For the numbers $E_m^{(i)}$ the equalities

$$E_m^{(i)} = (-1)^m (1 + E^{(i+2)})_m, \qquad E_{4k}^{(2)} = E_{4k}^{(3)}, \qquad E_{4k+2}^{(1)} = -E_{4k+2}^{(4)},$$

hold, with $E_r^{(k)} = E_r^{(k-4)}$ for $k > 4$.

With the aid of these numbers the polynomials $A_{i,m}(x)$ can be written in the form

$$A_{i,4k+j}(x) = \frac{(x + E^{(i+j)})_{4k+j}}{(4k+j)!}.$$

The following extremal theorems hold:

**Theorem 1.** Of all polynomials $P_m(x) \in L_{2,m-s}^{(i)}$ of the form

$$P_m(x) = \sum_{k=m-s}^{m} \sigma_k x^k + \sum_{k=0}^{m-s-1} p_k x^k,$$

where $\sigma_k$ $(k = m-s, \dots, m)$ are fixed, the polynomial least deviating from 0 on $[0,1]$ is

$$P_m^*(x) = \sum_{r=0}^{s} (m-r)!\, a_{m-r} A_{i,m-r}(x),$$

in which

$$a_{m-r} = \begin{cases} \sigma_{m-r}, & \text{for } r \equiv m+i+3,\ m+i \pmod 4, \\ \sum\limits_{k=0}^{r} C_{m-k}^{r-k} \sigma_{m-k}, & \text{for } r \equiv m+i+1,\ m+i+2 \pmod 4. \end{cases}$$

Moreover, the least deviation is determined by the formula

$$L_m^{(i)} = \left| \sum_{r=0}^{s} (-1)^{\alpha r} a_{m-r} E_{m-r}^{(j)} \right|,$$

where

$$\alpha = \frac{i^2 - i + 2}{2}, \qquad j \equiv m - r - \frac{3 + (-1)^i}{2} \pmod 4.$$

**Remark 1.** The coefficients $a_{m-r}$ are found as the solution of the system

$$\sigma_{m-r} = \sum_{k=0}^{r} C_{m-k}^{r-k} E_{r-k}^{(m+i-k)} a_{m-k} \qquad (r = 0, 1, \dots, s).$$

**Remark 2.** For the set of polynomials under consideration to be nonempty, it is necessary and sufficient that the coefficients $\sigma_k$ be such that the polynomial

$$\sum_{k=m-s}^{m} \frac{k!}{(k-m+s)!} \sigma_k x^{k-m+s}$$

does not change sign on $[0, 1]$.

**Remark 3.** From the theorem just formulated, in particular, for $s = 0$ there follows the extremal assertion of S. N. Bernstein from paper (3) (p. 548).

**Theorem 2.** Among all polynomials $P_m(x) \in {}^{(i)}_{2,s+1}$ of the form

$$P_m(x) = \sum_{k=s+1}^{m} \rho_k x^k + \sigma_s x^s + \sum_{k=0}^{s-1} \rho_k x^k,$$

where $\sigma_s$ is fixed, the polynomial

$$P_m^*(x) = s! \sigma_s A_{i,s}(x)$$

deviates least from 0 on $[0, 1]$, and the least deviation is determined by the formula

$$L_m^{(i)} = |\sigma_s E_s^{(i)}|,$$

where

$$j \equiv s - 1 \pmod 4 \quad \text{for } i = 1, 3; \qquad j \equiv s - 2 \pmod 4 \quad \text{for } i = 2, 4.$$

**Theorem 3.** Among all polynomials $P_m(x) \in {}^{(i)}_{2,m-1}$ of the form

$$P_m(x) = \rho_m x^m + \sigma_{m-1} x^{m-1} + \sum_{k=0}^{m-2} \rho_k x^k,$$

where $\sigma_{m-1}$ is fixed, the polynomial

$$P_m^*(x) = \begin{cases} (m-1)!\,\sigma_{m-1}[-A_{i,m}(x) + A_{i,m-1}(x)], & \text{for } m+i \equiv 1,2 \pmod 4, \\ -(m-1)!\,\sigma_{m-1}A_{i,m}(x), & \text{for } m+i \equiv 0,3 \pmod 4, \end{cases}$$

deviates least from 0 on $[0,1]$, and the magnitude of the least deviation is determined by the formula

$$L_m^{(i)} = \begin{cases} \left|\sigma_{m-1}\left[(-1)^{\beta}\dfrac{E_m^{(j+1)}}{m} + E_m^{(j)}\right]\right|, & \text{for } m+i \equiv 1,2 \pmod 4, \\ \left|\sigma_{m-1}\dfrac{E_m^{(j+1)}}{m}\right|, & \text{for } m+i \equiv 0,3 \pmod 4, \end{cases}$$

where

$$\beta = \frac{i^2 - i}{2}, \qquad j \equiv m - \frac{5 + (-1)^i}{2} \pmod 4.$$

**Remark.** Theorem 3 is obviously not a special case of Theorem 2 corresponding to $s = m-1$, since in Theorem 3 the extremal polynomial is sought in the class ${}^{(i)}_{2,m-1}$, which is broader than the class ${}^{(i)}_{2,m}$.

Received
20 X 1959

## CITED LITERATURE

[1] S. N. Bernstein, *Collected Works*, 1, No. 32, Publishing House of the Academy of Sciences of the USSR, 1952.
[2] S. N. Bernstein, *Collected Works*, 2, No. 100, Publishing House of the Academy of Sciences of the USSR, 1954.
[3] S. N. Bernstein, *Collected Works*, 2, No. 106, Publishing House of the Academy of Sciences of the USSR, 1954.