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Fig. 1

Figure 1: Fig. 1

Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE SYNTHESIS OF SELF-CORRECTING CONTACT CIRCUITS

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Let $\mathfrak{U}$ be a set of control systems in the sense of (^{1}). With each control system U_α there are associated a circuit Σ_{i_α} and a function Φ_{l_α}. The circuit characterizes the structure of the control system, and the function its behavior. In a number of cases the function Φ_{l_α} is determined by the circuit Σ_{i_α} in a unique way. Then $\Phi_{l_\alpha} = F(\Sigma_{i_\alpha})$, and we are entitled to say that the circuit Σ_{i_α} realizes the function Φ_{l_α}. In this case the set $\mathfrak{U}$ is associated with the sets $\mathfrak{S} = \{\Sigma_i\}$ of circuits and $F = \{\Phi_l\}$ of functions in such a way that $\Phi = F(\Sigma)$.

Suppose that, owing to certain circumstances, circuits Σ may "pass into a faulty" state Σ', so that $\Sigma' \in \mathfrak{S}$. Then with each circuit Σ there will be associated a subset $\mathfrak{S}_\Sigma$ of all circuits Σ' representing faulty states of the circuit Σ. It may be assumed that this correspondence is specified by a function G, i.e., $\mathfrak{S}_\Sigma = G(\Sigma)$.

Definition. A circuit Σ is called **self-correcting for** G if, whatever $\Sigma' \in \mathfrak{S}_\Sigma$ may be, $F(\Sigma') = \Phi$, where $\Phi = F(\Sigma)$.

Fig. 1

In other words, under any fault determined by the function G, the circuit Σ' realizes the same function as the circuit Σ.

The question is whether there exist circuits self-correcting for G.

In order to approach the solution of the question posed, let us turn to a model object. Thus, let $\mathfrak{U}$ be the set of all contact circuits realizing functions of the algebra of logic. Here $\mathfrak{S}$ consists of contact circuits, and F of all functions of the algebra of logic.

Suppose that in every contact circuit Σ, at any moment of time, no more than m contacts can "fail." More precisely: some of these contacts have short-circuited, while the remaining ones have opened in such a way that the relay is unable to change their position. The circuit Σ' thus obtained represents a faulty state of the circuit Σ. Thus, $\mathfrak{S}_\Sigma$ is the set of all circuits obtained from Σ by the occurrence of faults of the indicated type.

In this case, as is not difficult to see, for every function $\Phi(x_1, \dots, x_n)$ one can construct a self-correcting contact circuit. For this, in the circuit Σ realizing the function Φ, one should replace each contact $\bullet\!-x^\sigma-\!\bullet$ by the subcircuit shown in Fig. 1a (the contact is duplicated $m+1$ times in parallel and $m+1$ times in series).

However, this solution leads to a complication of the circuit by a factor of $(m+1)^2$, which makes it unsuitable. In connection with this, let us refine the question posed.

Let $L(\Sigma)$ denote the complexity of a circuit. Denote by $L(\Phi)$ the quantity $\inf L(\Sigma)$, where the lower bound is taken over all circuits realizing Φ. Suppose that there exist circuits self-correcting for G that realize the function Φ. Let $L_G(\Phi) = \inf L(\Sigma)$, where the lower bound is taken over all self-correcting circuits realizing Φ. The question is: how much does $L_G(\Phi)$ differ from $L(\Phi)$? Or does the construction of self-correcting circuits require an essential complication of the circuits?

To resolve this question, let us return again to our model object. Here let $L(\Sigma)$ denote the number of contacts in the circuit Σ. We have

$$L(\Phi) \leqslant L_G(\Phi) \leqslant (m+1)^2 L(\Phi),$$

or, if we pass to the class of functions depending on n variables and put $L(n) = \max L(\Phi)$ and $L_G(n) = \max L_G(\Phi)$,

$$L(n) \leqslant L_G(n) \leqslant (m+1)^2 L(n).$$

Thus, the problem that interests us now consists in estimating the growth of the function $L_G(n)$. However, even in this variant the problem contains considerable difficulties. In view of this, it was necessary to go to a further narrowing of the problem, taking $m = 1$ and restricting ourselves to faults of the type of short circuits. It is easy to see that, in this case, in the presence of a short circuit only new circuits with nonzero conductivity can arise. The appearance of such circuits is possible, first, due to shorting in circuits with nonzero conductivity and, second, due to shorting in circuits with zero conductivity. The first result, under certain restrictive assumptions, was obtained by Yu. G. Potapov. In the present note another solution is proposed, not relying on any hypotheses.

Let the function $L_3^m(n)$ characterize the complexity of circuits self-correcting for short circuits in m contacts.

Theorem 1. $L_3^1(n) \sim 2^n/n$.

Table 1 diagram: a schematic decomposition of the truth table for $f(\sigma_1, \dots, \sigma_n)$, with variables $x_1, \dots, x_k$ along one block, variables $x_{k+1}, \dots, x_n$ along another, rows grouped into strips of size s and a final strip of size s'.

Figure 2: Table 1 diagram: a schematic decomposition of the truth table for $f(\sigma_1, \dots, \sigma_n)$, with variables $x_1, \dots, x_k$ along one block, variables $x_{k+1}, \dots, x_n$ along another, rows grouped into strips of size s and a final strip of size s'.

Proof. We shall indicate a method which makes it possible, for every function $f(x_1, \dots, x_n)$, to construct a self-correcting circuit for one short circuit and having asymptotically $2^n/n$ contacts. For this we take a circuit constructed by the method of O. B. Lupanov (^{2}), realizing the function f, and show how it must be altered in order to obtain the required self-correcting circuit.

Table 1

Divide the table (Table 1), which defines the function f, into strips, each of which (except, perhaps, the last) contains exactly s rows, and the last contains s' ($s' \leqslant s$) rows. Let p be the number of strips ($p \leqslant [2^k/s] + 1$). Within each strip divide the columns into groups, so that all identical columns enter one group. Let $f_{ij}(x_1, \dots, x_n)$ be the function that coincides with f on the j-th group of the i-th strip and is equal to 0 on the remaining sets. Obviously,

$$f_{ij}(x_1, \dots, x_n) = f_{ij}^{(1)}(x_1, \dots, x_k) \,\&\, f_{ij}^{(2)}(x_{k+1}, \dots, x_n).$$

Let

$$\bigvee_j f_{ij}^{(1)} f_{ij}^{(2)} = f_i, \qquad \text{then} \qquad \bigvee_{i=1}^{p} f_i(x_1, \dots, x_n) = f(x_1, \dots, x_n).$$

Take r equal to a power of two and smaller than $n - k$. Then the r-dimensional cube with axes $x_{k+1}, \dots, x_{k+r}$ can be partitioned into $2^r/r$ nonintersecting spheres of radius 1 (in the sense of the Hamming metric (^{3})) with centers $(\beta_{k+1}^h, \dots, \beta_{k+r}^h)$, where $h = 1, \dots, 2^r/r$.

The circuit $\mathfrak{A}'$, realizing the function f (without the requirement of self-correction), as is known (^{2}), is constructed by parallel connection of circuits $\mathfrak{A}_i'$ realizing f_i. We describe the structure of the circuits $\mathfrak{A}_i'$. For this purpose we construct a system of $[1, q]$-poles $M_1', \dots, M_6'$ such that each subsequent one is obtained by completing the preceding one, and $M_6' = \mathfrak{A}_i'$ (Fig. 2).

M_1' is a $[1, 2^r]$-pole representing a contact tree in the variables $x_{k+1}, \dots, x_{k+r}$. M_2' is a $[1, 2^r/r]$-pole obtained from M_1' by combining r outputs corresponding to the points of one and the same sphere. M_2' realizes the characteristic functions of the spheres $\varphi_h(x_{k+1}, \dots, x_{k+r})$ ($1 \leq h \leq 2^r/r$). M_3' is a $[1, 2^{n-k}/r]$-pole

Fig. 2

Figure 3: Fig. 2

obtained from M_2' by connecting to each of the outputs contact trees in the variables $x_{k+r+1}, \dots, x_n$. M_3' realizes functions of the form

$$\varphi_h(x_{k+1}, \dots, x_{k+r}) x_{k+r+1}^{\sigma_{k+r+1}} \cdots x_n^{\sigma_n}.$$

M_4' is a $[1, 2^{n-k}]$-pole obtained from M_3' by connecting to the outputs corresponding to the sphere with center $(\beta_{k+1}^h, \dots, \beta_{k+r}^h)$ a $[1, r]$-pole shown in Fig. 3a $(h = 1, 2, \dots, 2^r/r)$. M_4' realizes all conjunctions of the form

$$x_{k+1}^{\sigma_{k+1}} \cdots x_n^{\sigma_n}.$$

M_5' is a multipole obtained from M_4' by combining (inside each sphere) certain outputs so that M_5' realizes functions of the form

$$f_{ijh}^{(2)}(x_{k+1}, \dots, x_n) = f_{ij}^{(2)}(x_{k+1}, \dots, x_n) \,\&\, \varphi_h(x_{k+1}, \dots, x_{k+r})$$
$$(j = 1, 2, \dots; \quad 1 \leq h \leq 2^r/r).$$

M_6' is a $[1, 1]$-pole obtained from M_5' by connecting to its outputs π-circuits and then combining all outputs into one. Namely, to the output corresponding to the function $f_{ijh}^{(2)}$ there is connected a π-circuit corresponding to the perfect disjunctive normal form of the function $f_{ij}^{(1)}(x_1, \dots, x_k)$. The multipole M_6' realizes the function $f_i(x_1, \dots, x_n)$.

Fig. 2

If in this construction we take $r = 2^{[1/2 \log_2 n]}$, $k = [2 \log_2 n]$, and $s = [n - 2\sqrt{n}]$, then we obtain the asymptotic inequality $L(n) \lesssim 2^n/n$.

The circuit constructed by us has one property which is used in the subsequent part of the proof. Let the circuit $\mathfrak{A}'$ be partitioned into the direct sum of three subcircuits $\mathfrak{A}_1'$, $\mathfrak{A}_2'$, and $\mathfrak{A}_3'$, where $\mathfrak{A}_1'$ consists of all multipoles M_3', $\mathfrak{A}_2'$ consists of all multipoles $M_5' \setminus M_3'$, and $\mathfrak{A}_3'$ consists of all multipoles $M_6' \setminus M_5'$. Let $L(\mathfrak{A}_1')$, $L(\mathfrak{A}_2')$, and $L(\mathfrak{A}_3')$ be the numbers of contacts in the corresponding subcircuits. It is easy to see that, for the above values of the parameters r, k, and s, the relations

$$L(\mathfrak{A}_1') = o(L(\mathfrak{A}_2')), \qquad L(\mathfrak{A}_3') = o(L(\mathfrak{A}_2')). \tag{1}$$

The latter means that $L(\mathfrak{A}_2') \sim 2^n/n$, i.e., that the subscheme $\mathfrak{A}_2'$ contains almost all the contacts of the circuit $\mathfrak{A}'$.

We now turn to the description of those changes in the constructed circuit which make it possible to obtain from it a self-correcting circuit satisfying the condition of the theorem. To this end, in the subschemes $\mathfrak{A}_1'$ and $\mathfrak{A}_3'$ each

Fig. 3

Figure 4: Fig. 3

contact x^{σ} should be duplicated in series, as indicated in Fig. 1b. We denote the resulting subschemes by $\mathfrak{A}_1$ and $\mathfrak{A}_3$.

For the subscheme $\mathfrak{A}'$ we shall give a special construction. Recall that, in the synthesis of the multipoles M_5, we performed, in a certain way, identifications of the outputs of the multipoles M_4', and, moreover, identifications of this kind were made only within the individual spheres of each multipole. Consider one of the identifications made for the outputs of the sphere with center $(\beta^{h}_{k+1}, \dots, \beta_{k+r})$, and replace it by a certain special subscheme. For this purpose we take an $[r, 1]$-pole (Fig. 3b) and connect to its l-th input ($l = 1, 2, \dots, r$) all the outputs of the multipole M_4' which participated in the identification under consideration and which correspond to the contact $x^{\overline{h}}_{k+l}$ (Fig. 3b); the output of this multipole is connected with the corresponding π-circuit (Fig. 2). We carry out such a construction for all identifications. We denote the subscheme obtained in this way by $\mathfrak{A}_2$.

Fig. 3

Thus, we have constructed the circuit

$$\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3.$$

It is easy to see that it realizes the same function $f(x_1, \dots, x_n)$. The fact that this circuit is self-correcting follows from the fact that in every chain of the circuit $\mathfrak{A}$ having nonzero conductance, each contact occurs twice, and in every chain having zero conductance there are at least two pairs of opposite contacts.

Let us estimate the complexity $L(\mathfrak{A})$ of the circuit $\mathfrak{A}$. From (1) and the relations

$$L(\mathfrak{A}_1) = 2L(\mathfrak{A}_1'), \quad L(\mathfrak{A}_3) = 2L(\mathfrak{A}_3'), \quad L(\mathfrak{A}_2) \leqslant L(\mathfrak{A}_2') + \sqrt{n}\, 2^{n-2\sqrt{n}}$$

it follows that $L(\mathfrak{A}) \lesssim 2^n/n$. Thus,

$$L_3^1(n) \leqslant L(\mathfrak{A}) \lesssim 2^n/n.$$

The lower estimate for $L_3^1(n)$ follows from (4). The theorem is proved.

With a slight complication of this synthesis method it is not difficult to obtain the following result:

Theorem 2.

$$L_3^m(n) \lesssim \left(\left[\frac{m}{2}\right] + 1\right) \frac{2^n}{n}.$$

Theorem 2 gives, for almost all functions, self-correcting circuits that are twice as simple as those obtained by the trivial method.

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Note: Figure translations are in progress. See original paper for figures.