MATHEMATICS

FLORIN CONSTANTINESCU

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Abstract

Full Text

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FLORIN CONSTANTINESCU

ON CHEBYSHEV SETS

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1. Let X_n be an n-dimensional Banach space, and let M be some set in X_n. As usual, we shall call the distance from a point $x \in X_n$ to the set M the number
$$\rho(x, M) = \inf_{y \in M} \|x - y\|.$$

Together with N. V. Efimov and S. B. Stechkin ([1]), we shall call the set M: 1) a **set of existence**, if for every $x \in X_n$ there is a nearest element $y_0 \in M$ such that $\rho(x, y) = \rho(x, y_0)$; 2) a **Chebyshev set**, if for every $x \in X_n$ there is a unique element $y_0 \in M$ such that $\rho(x, M) = \rho(x, y_0)$.

Obviously, every closed bounded set is a set of existence, and every set of existence (in particular, every Chebyshev set) is a closed set.

N. V. Efimov and S. B. Stechkin proved the following theorem:

In an n-dimensional Banach space X_n, the class of bounded Chebyshev sets coincides with the class of bounded closed convex sets if and only if the unit sphere of X_n is strictly convex and has no conical points.

2. We wish to prove that the theorem remains valid if the boundedness requirement is omitted from it.

Let M be such a set. We shall first prove that M is a set of existence.

For this purpose consider the sphere $S(x, r + a)$ with center x and radius $r + a$, where $r = \rho(x, M)$ and $a > 0$. Then the set $P_a = M \cap S(x, r + a)$ is nonempty, bounded, and closed. Hence it follows that there exists an element $y_0 \in P_a$ for which we have $\rho(x, P_a) = \rho(x, y_0)$. On the other hand, whatever element $y \in M$ lying outside the sphere $S(x, r + a)$ may be, we have $\rho(x, y) > r + a$. Consequently, $\rho(x, M) = \rho(x, P_a)$, i.e. there exists an element $y_0 \in M$ such that $\rho(x, M) = \rho(x, y_0)$.

Theorem 1. *If in an n-dimensional Banach space X_n the unit sphere has no conical points, then every Chebyshev set is convex.*

Let M be a Chebyshev set, $x \notin M$. Then there exists a unique element for which $\rho(x, M) = \rho(x, y_0)$. Consider the spheres with center at x, whose radii are the numbers $\rho(x, M) + 1, \rho(x, M) + 2, \dots, \rho(x, M) + n, \dots$. Forming the intersections of these spheres with the set M, we obtain nonempty sets $P_1, P_2, \dots, P_n, \dots$.

The sets P_n are Chebyshev sets. Indeed, since the spheres are bounded, closed, and convex sets, on the basis of Theorem 2 of N. V. Efimov and S. B. Stechkin it follows that these spheres are Chebyshev sets. Taking into account that the intersection of two Chebyshev sets is also a Chebyshev set and applying the aforementioned theorem of Efimov and Stechkin, we obtain that the sets P_n are convex. On the other hand, $P_1 \subset P_2 \subset \cdots$

$\cdots \subset P_n \subset \ldots$ and $M = \bigcup_{n=1}^{\infty} P_n$; hence it follows at once that the set M is also convex. The theorem is proved.

3. If the sphere X_n is strictly convex, then it is not difficult to prove that every convex set is Chebyshev.

It is easy to see that in a space X_n in which the unit sphere is not strictly convex, there exist convex non-Chebyshev sets; and also that in a space X_n in which the unit sphere contains conical points, there exist Chebyshev sets that are not convex.

Hence, and on the basis of Theorem 1, we obtain Theorem 2:

Theorem 2. *In an n-dimensional Banach space X_n, the class of Chebyshev sets coincides with the class of convex sets if and only if the unit sphere is strictly convex and has no conical points.*

State University
named after V. Babeș
Cluj, Romania

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REFERENCES

1. N. V. Efimov, S. B. Stechkin, DAN, 118, No. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.