MATHEMATICS

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Abstract

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ON THE THEORY OF SUMMATION OF DOUBLE SERIES BY BOREL METHODS

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In the works of V. G. Chelidze and V. A. Berekashvili (^{1,2}), the Borel summation of double series is considered, defined as follows:

Let a double series be given

$$\sum_{i,k=0}^{\infty} a_{ik} \tag{1}$$

and its partial sums

$$A_{mn} = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik}.$$

Suppose that the double power series

$$A(x, y) = \sum_{i,k=0}^{\infty} A_{ik} \frac{x^i y^k}{i!k!}$$

converges for all values $x \geq 0$ and $y \geq 0$.

The series (1) is called B_λ-summable to the sum S if

$$\lim_{(x,y)_\lambda \to \infty} e^{-(x+y)} A(x, y) = S.$$

By the symbol $(x, y)_\lambda \to \infty$ we understand tending to infinity inside the sector defined by the inequalities

$$\lambda \leq \frac{y}{x} \leq \frac{1}{\lambda}, \qquad 0 < \lambda < 1.$$

We call the series (1) B-summable to the sum S if

$$\lim_{x\to\infty,\ y\to\infty} e^{-(x+y)}A(x,y) = S.$$

The latter means that for every $\varepsilon > 0$ there exists a number $T(\varepsilon)$ such that

$$\left|e^{-(x+y)}A(x,y) - S\right| < \varepsilon$$

as soon as $x > T$ and $y > T$.

V. G. Chelidze (^{1}) proved the following proposition:

If the double series (1) converges and has sum S, and the partial sums A_{mn} of this series satisfy the condition

$$|A_{mn}| \leq M(m+1)^{\alpha}(n+1)^{\beta}, \tag{2}$$

where M, α, β are positive numbers independent of m and n, then the double series (1) is B_λ-summable to the sum S for all values of the number λ $(0 < \lambda < 1)$.

V. A. Berekashvili showed (^{2}) that condition (2) can be replaced by the weaker one, namely

$$|A_{mn}| \leqslant M a^{o(n)+o(n)}, \tag{3}$$

where M and a are positive numbers independent of m and n.

We have succeeded in proving the regularity of the B_λ-method under more general conditions than those of V. G. Chelidze and V. A. Berekashvili.

Theorem 1. *Let the double series (1) converge, have sum S, and moreover satisfy two conditions:*

$$\left|\sum_{k=0}^{\infty} A_{ik}\frac{y^k}{k!}\right| \leqslant M_i e^{(1+\lambda')y}, \tag{4}$$

$$\left|\sum_{i=0}^{\infty} A_{ik}\frac{x^i}{i!}\right| \leqslant N_k e^{(1+\lambda')x}, \tag{5}$$

where $M_i, N_k, \lambda' < 1$ are positive numbers independent of x and y. Then the series (1) is B_λ-summable to the sum S for $\lambda > \lambda'$.

Let the double power series

$$a(x, y) = \sum_{i,k=0}^{\infty} a_{ik} \frac{x^i y^k}{i!k!}$$

converge for all values of x and y (i.e., $a(x, y)$ is an entire function). Put

$$\Phi(x, y) = \int_0^x \int_0^y e^{-(t+\tau)} a(t, \tau)\, dt\, d\tau.$$

The series (1) is called B'_λ-summable to the sum S if

$$\lim_{(x,y)_\lambda \to \infty} \Phi(x, y) = S.$$

If

$$\lim_{x\to\infty,\ y\to\infty} \Phi(x, y) = S,$$

then we call the series (1) B'-summable to the sum S.

The methods B and B' are limiting cases of the methods B_λ and B'_λ as $\lambda \to 0$. Restricting ourselves to the consideration of such double series (1) for which the functions $a(x, y)$ and $A(x, y)$ are simultaneously entire functions, we prove the following theorems:

Theorem 2. *The methods B_λ and B'_λ are equivalent if and only if the relation*

$$\lim_{(x,y)_\lambda \to \infty} \left[\Phi''_{xy}(x, y) + \Phi'_x(x, y) + \Phi'_y(x, y)\right] = 0$$

holds.

An analogous assertion holds for the methods B and B'.

This theorem is an analogue of a well-known theorem in the theory of Borel summation of simple series (([3]), p. 229, Theorem 123).

Theorem 3. *If the function $\Phi(x, y)$, defined in the domain $x \geqslant 0,\ y \geqslant 0$, has continuous first partial derivatives and a continuous mixed derivative of second order and satisfies the conditions:*

$$1) \quad \Phi(x, 0) = O(e^{\lambda x}) \quad \text{as } x \to \infty;$$

$$\Phi(0, y) = O(e^{\lambda y}) \quad \text{as } y \to \infty, \quad 0 < \lambda < 1;$$

2)
$$\left|\Phi''_{xy}(x,y)+\Phi'_x(x,y)+\Phi'_y(x,y)+\Phi(x,y)\right|\le Me^{\lambda(x+y)}$$
for all $x\ge 0,\ y\ge 0$;

3)
$$\lim_{x\to\infty,\ y\to\infty}\left[\Phi''_{xy}(x,y)+\Phi'_x(x,y)+\Phi'_y(x,y)+\Phi(x,y)\right]=S,$$
then
$$\lim_{(x,y)_{\lambda+\delta}\to\infty}\Phi(x,y)=S\quad\text{for } 0<\delta<1-\lambda.$$

Proof. Without loss of generality, one may assume that $S=0$. Putting
$$\Phi''_{xy}(x,y)+\Phi'_x(x,y)+\Phi'_y(x,y)+\Phi(x,y)=\varepsilon(x,y),$$
we obtain
$$\frac{\partial^2}{\partial x\,\partial y}\left[e^{x+y}\Phi(x,y)\right]=e^{x+y}\varepsilon(x,y);$$
therefore,
$$\Phi(x,y)=e^{-y}\Phi(x,0)+e^{-x}\Phi(0,y)-e^{-(x+y)}\Phi(0,0)+e^{-(x+y)}\int_0^x\int_0^y e^{t+\tau}\varepsilon(t,\tau)\,dt\,d\tau.$$

Using the hypotheses of the theorem and the last relation, it is easy to show that
$$\lim_{(x,y)_{\lambda+\delta}\to\infty}\Phi(x,y)=0.$$

This implies the validity of our theorem.

This theorem, in a simpler formulation, is known for functions of one variable (([3], p. 138, Theorem 53). As in the case of one variable (([3], p. 230, Theorem 124), it finds application in the theory of Borel summation.

Theorem 4. Let $0<\lambda'<\lambda<1$. If the double series (1) is summable by the method B to the value S and
$$|A(x,y)|\le Me^{(1+\lambda')\lambda(x+y)},$$
where M is a positive number independent of x and y, then the series (1) is summable by the method B'_λ to the value S.

For the proof it suffices to establish the relation
$$e^{-(x+y)}A(x,y)=\Phi''_{xy}(x,y)+\Phi'_x(x,y)+\Phi'_y(x,y)+\Phi(x,y),$$
and then to use the preceding theorem.

Remark. With respect to double series satisfying the condition of Theorem 4, the method B'_λ is stronger than the method B. Bearing in mind an example of a single series summable by the method B' and not summable by the method B (([3], p. 230), it is easy to construct a double series summable by the method B'_λ and not summable by the method B.

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[3] G. Hardy, *Divergent Series*, Moscow, 1951.

Note: Figure translations are in progress. See original paper for figures.

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