# STABILITY OF A CYLINDRICAL SHELL REINFORCED BY RINGS UNDER EXTERNAL PRESSURE

Fig. 1

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## Abstract

## Full Text

## THEORY OF ELASTICITY

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# STABILITY OF A CYLINDRICAL SHELL REINFORCED BY RINGS UNDER EXTERNAL PRESSURE

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The problem is solved of the stability of a circular cylindrical shell reinforced by rings, with hinged edges, under the action of external pressure. The known solutions of this practically important problem are not entirely satisfactory. Some of them were obtained in a very approximate way by the energy method; others are based on replacing a ribbed isotropic shell by a smooth anisotropic shell (such a replacement is associated with a certain indeterminacy and with the condition that the number of rings be sufficiently large and that the rings themselves be sufficiently flexible). Often these solutions do not agree with experimental results. To obtain a more complete solution, the method set forth in ($^1$)* is used here. By means of sections perpendicular to the axis of the shell, the rings are separated from the adjacent portions of the shell, and these elements are considered in isolation from one another. Forces and moments are applied to them that characterize their interaction at the proper value of the pressure. These forces and moments, like the displacements, are represented in the form of sums of their fundamental and additional values. For simplicity, the rings are considered identical, uniformly distributed along the shell, and having a rectangular cross section (see Fig. 1).

**Fig. 1**

Let $m$ be the number of portions (spans) of the shell, and, consequently, $(m-1)$ the number of rings. For the additional radial displacement $w_i$ of the $i$-th portion of the shell we have the equation (see ($^1$), (6))

$$\varepsilon\left(\frac{\partial^8 w_i}{\partial\varphi^8}+2\frac{\partial^6 w_i}{\partial\varphi^6}+\frac{\partial^4 w_i}{\partial\varphi^4}\right)+\frac{\partial^4 w_i}{\partial\xi^4}+\frac{qR}{Eh}\left(\frac{\partial^6 w_i}{\partial\varphi^6}+\frac{\partial^4 w_i}{\partial\varphi^4}\right)=0. \qquad (1)$$

(The notation introduced in ($^1$), which retains its meaning, is not explained; the coordinate $\xi$ is measured from the middle of the shell span.) For the corresponding displacements $u_i, v_i$ in the axial and circumferential directions, the equations (4), (5), (7) of paper ($^1$) are valid, with $u, v, w$ replaced in them by $u_i, v_i, w_i$. For the $i$-th ring we have the equation (see ($^1$), (10))

$$\frac{EI}{R^2}\frac{d}{d\varphi}\left(\frac{d^2}{d\varphi^2}+1\right)\left(\frac{d^2\tilde{u}_i}{d\varphi^2}+\frac{d\widetilde{w}_i}{d\varphi}\right)+kaRq\frac{d}{d\varphi}\left(\frac{d^2\tilde{u}_i}{d\varphi^2}+\frac{d\widetilde{w}_i}{d\varphi}\right)-R^2\left(\frac{dX_i}{d\varphi}-Z_i\right)+\left(\frac{d^2}{d\varphi^2}+1\right)RK_i'=0, \tag{2}$$

where $\tilde{u}_i$, $\widetilde{w}_i$ are the additional displacements of the $i$-th ring, corresponding to the displacements $\tilde{u}$ and $\widetilde{w}$ in paper ($^1$),

$$X_i=-N_1^{(i)}\big|_{\xi=\rho/2}+N_1^{(i+1)}\big|_{\xi=-\rho/2}, \qquad Z_i=-S^{(i)}\big|_{\xi=\rho/2}+S^{(i+1)}\big|_{\xi=-\rho/2},$$
$$K_i=-H^{(i)}\big|_{\xi=\rho/2}+H^{(i+1)}\big|_{\xi=-\rho/2};$$

$$\begin{aligned} N_1^{(i)}&=-\left[\frac{Eh^3}{12(1-\nu^2)R^3}\right]\left(\frac{\partial^3 w_i}{\partial\xi\,\partial\varphi^2}+\frac{\partial^2 v_i}{\partial\xi\,\partial\varphi}\right),\\ S^{(i)}&=\left[\frac{Eh}{2(1+\nu)R}\right]\left(\frac{\partial u_i}{\partial\varphi}+\frac{\partial v_i}{\partial\xi}\right),\\ H^{(i)}&=\left[\frac{Eh^3}{12(1+\nu)R^2}\right]\left(\frac{\partial^2 w_i}{\partial\xi\,\partial\varphi}+\frac{\partial v_i}{\partial\xi}\right)\end{aligned} \tag{3}$$

($\rho R$ is the span length between adjacent rings).

Using the compatibility conditions $\tilde{u}_i=w_i\big|_{\xi=\rho/2}$, $\widetilde{w}_i=v_i\big|_{\xi=\rho/2}$ and the equations for $u_i, v_i$, from (2) and (3), similarly to the way in which equation (13) was obtained in (1), we obtain the following equation:

$$\begin{aligned}&\left[\frac{H^3}{12hR^2}\frac{\partial^4}{\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)^2 w_i+\frac{kRq}{Eh}\frac{\partial^4}{\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w_i\right.\\ &\quad-\frac{2-\nu}{12(1-\nu^2)}\frac{h^2}{Ra}\frac{\partial^5}{\partial\xi\,\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w_i-\frac{h^2}{12(1-\nu^2)Ra}\frac{\partial^3}{\partial\xi\,\partial\varphi^2}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w_i\\ &\quad\left.-\frac{R}{a}\frac{\partial^3 w_i}{\partial\xi^3}\right]_{\xi=\rho/2}+\left[\frac{2-\nu}{12(1-\nu^2)}\frac{h^2}{Ra}\frac{\partial^5}{\partial\xi\,\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w_{i+1}\right.\\ &\quad\left.+\frac{h^2}{12(1-\nu^2)Ra}\frac{\partial^3}{\partial\xi\,\partial\varphi^2}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w_{i+1}+\frac{R}{a}\frac{\partial^3 w_{i+1}}{\partial\xi^3}\right]_{\xi=-\rho/2}=0,\end{aligned} \tag{4}$$

where $k$, in comparison with (1), has the new value: $k=1+(1+h/H)/[h/H+\varkappa a/2R]$ (it is assumed that $\rho\gg h/R$). Put $w_i=$

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* In ($^1$), misprints crept into equation (3): instead of $L_1^{(6)}(w)$ and $\frac{\partial^2}{\partial\xi^2}L_2^{(2)}(w)$ there should be $L_1^{(8)}(w)$ and

$C_i \cos(\mu\xi + \alpha_i) \sin n\varphi$. The boundary conditions for the shell edges $w_1\big|_{\xi=-\rho/2} = w_m\big|_{\xi=\rho/2} = [\partial^2 w_1/\partial\xi^2]_{\xi=-\rho/2} = [\partial^2 w_m/\partial\xi^2]_{\xi=\rho/2} = 0$ will be satisfied if $\cos(-\mu\rho/2+\alpha_1) = \cos(\mu\rho/2+\alpha_m) = 0$, whence $\alpha_1 = \mu\rho/2 - \pi/2$, $\alpha_m = \pi/2 - \mu\rho/2 = -\alpha_1$. Since $w_i\big|_{\xi=\rho/2} = w_{i+1}\big|_{\xi=-\rho/2}$, then

$$C_i \cos(\mu\rho/2 + \alpha_i) = C_{i+1} \cos(-\mu\rho/2 + \alpha_{i+1}) \quad (i = 1, 2, \dots, m-1). \tag{5}$$

(We ignore equality of the rotation angles of the elements being joined; on this point see (1).) Substituting the adopted expression for $w_i$ into equations (1), (4) and using (5), we obtain

$$qR/Eh = \varepsilon(n^2 - 1) + \mu^4/n^4(n^2 - 1), \tag{6}$$

$$\frac{qR}{Eh} = \frac{H^3(n^2-1)}{12khR^2} - \mu \left[\frac{h^2(2-\nu-n^{-2})}{12(1-\nu^2)kaR} + \frac{R}{ka}\frac{\mu^2}{n^4(n^2-1)}\right]\left[\operatorname{tg}\left(\frac{\mu\rho}{2}+\alpha_i\right) + \operatorname{tg}\left(\frac{\mu\rho}{2}-\alpha_{i+1}\right)\right] \quad (i = 1, 2, \dots, m- \tag{7}$$

For fixed $n$, equalities (6), (7) constitute a system of $m$ equations with $m$ unknowns: $q, \mu, \alpha_2, \alpha_3, \dots, \alpha_{m-1}$. For the smallest value $m = 2$, instead of equalities (7) we shall have one equality which, by virtue of the condition $\alpha_m = \alpha_2 = -\alpha_1 = \pi/2 - \mu\rho/2$, takes the form

$$\frac{qR}{Eh} = \frac{H^3(n^2-1)}{12khR^2} + \mu \left[\frac{h^2(2-\nu-n^{-2})}{6(1-\nu^2)kaR} + \frac{2R}{ka}\frac{\mu^2}{n^4(n^2-1)}\right] \operatorname{ctg}\mu\rho. \tag{7'}$$

From equalities (7) and the equality $\alpha_m = -\alpha_1$ it follows that $\alpha_{m-j} = -\alpha_{j+1}$. Therefore, for $m > 2$ one may consider not all equations (7), but only those that correspond to the values $i = 1, 2, \dots, p$, if $m = 2p$, or $i = 1, 2, \dots, p-1$, if $m = 2p - 1$. Bearing in mind that these equations must be solved jointly with (6), and using the equalities $\mu\rho/2 + \alpha_1 = \mu\rho - \pi/2$, $\operatorname{tg}(\mu\rho/2 - \alpha_i) = [\operatorname{tg}\mu\rho - \operatorname{tg}(\mu\rho/2+\alpha_i)]/[1 + \operatorname{tg}\mu\rho\,\operatorname{tg}(\mu\rho/2+\alpha_i)]$, one can replace equations (7), for $m > 3$, by the following:

$$\frac{qR}{Eh} = A(n) + B(\mu, n)\left[\operatorname{ctg}\mu\rho - \frac{\operatorname{tg}\mu\rho - \operatorname{tg}(\mu\rho/2+\alpha_2)}{1+\operatorname{tg}\mu\rho\,\operatorname{tg}(\mu\rho/2+\alpha_2)}\right], \tag{8}$$

$$\operatorname{tg}\left(\frac{\mu\rho}{2}+\alpha_i\right) = 2C(\mu,n) - \frac{\operatorname{tg}\mu\rho - \operatorname{tg}\left(\mu\rho/2+\alpha_{i+1}\right)}{1+\operatorname{tg}\mu\rho\,\operatorname{tg}\left(\mu\rho/2+\alpha_{i+1}\right)}$$

$$\begin{gathered}(i = 2, 3, \dots, p-1, \quad \text{if } m = 2p > 2; \\ i = 2, 3, \dots, p-2, \quad \text{if } m = 2p-1 > 3),\end{gathered} \tag{9}$$

$$\operatorname{tg}\left(\frac{\mu\rho}{2}+\alpha_p\right)=C(\mu,n)\quad(\text{if } m=2p>2),$$

$$\operatorname{tg}\left(\frac{\mu\rho}{2}+\alpha_{p-1}\right)=2C(\mu,n)-\operatorname{tg}\frac{\mu\rho}{2}\quad(\text{if } m=2p-1>3),$$

where

$$A(n)=\frac{H^3(n^2-1)}{12khR^2},\qquad B(\mu,n)=\mu\left[\frac{h^2(2-\nu-n^{-2})}{12(1-\nu^2)kaR}+\frac{R}{ka}\frac{\mu^2}{n^4(n^2-1)}\right],$$

$$C(\mu,n)=[1/2B(\mu,n)]\,[A(n)-\varepsilon(n^2-1)-\mu^4/n^4(n^2-1)].$$

For $m=3$ we obtain $\alpha_2=0$, and system (7) is replaced by a single equation

$$\frac{qR}{Eh}=\frac{H^3(n^2-1)}{12khR^2}+\mu\left[\frac{h^2(2-\nu-n^{-2})}{12(1-\nu^2)kaR}+\frac{R}{ka}\frac{\mu^2}{n^4(n^2-1)}\right]\left(\operatorname{ctg}\mu\rho-\operatorname{tg}\frac{\mu\rho}{2}\right). \tag{7”}$$

Thus, the solution of systems (6), (7), when $m=2$ or $m=3$, reduces to the solution of systems (6), (7′) or (6), (7″), and when $m>3$—to the solution of systems (6), (8), where $\operatorname{tg}(\mu\rho/2+\alpha_2)$ is determined from the recurrence formulas (9). The solution of these systems for any fixed value of $n$ ($n=2,3,\ldots$) determines an eigenvalue $q$. It is easy to find it graphically. For this purpose, for the selected value of $n$, one must construct the graphs of the quantity $qR/Eh$ as a function of $\mu$ according to formulas (6) and (8) (or (7′), if $m=2$) and determine the points of intersection of these two graphs. The values of $q$ corresponding to these points will be eigenvalues. Without analyzing how many points the two indicated graphs intersect in, denote the lowest of these points by $M_n$. Constructing, for a number of values of $n$, pairs of the indicated graphs, we determine $q_{\text{cr}}R/Eh$ as the ordinate of the lowest of the points $M_n$ ($q_{\text{cr}}$ is the critical value of $q$, i.e., the least of the eigenvalues). The expressions on the right-hand sides of (6), (7′), (7″), (8) are even functions of $\mu$ (as is evident from equalities (9), $\operatorname{tg}(\mu\rho/2+\alpha_2)$ is an odd function of $\mu$). Therefore it is possible to consider only positive values of $\mu$. For them the quantity $qR/Eh$, determined by formula (6), will be a monotonically increasing function of $\mu$ (for fixed $n$). Consequently, $M_n$ is that point of intersection of the two indicated graphs which corresponds to the smallest positive value of $\mu$. Since for a shell without rings the values of $\mu$ and $n$ corresponding to $q_{\text{cr}}$ decrease with increasing shell length, then for the problem under consideration the sought values of $\mu$ and $n$ (corresponding to the lowest of the points $M_n$, i.e., corresponding to $q_{\text{cr}}$) must, obviously, lie in the intervals

$$\underline{\mu} \le \mu \le \overline{\mu}, \qquad \underline{n} \le n \le \overline{n},$$

where $\underline{\mu} = \pi/m\rho$ and $\underline{n}$ correspond to $q_{\text{cr}}$ for the shell under consideration deprived of reinforcing rings, while $\overline{\mu} = \pi/\rho$ and $\overline{n}$ correspond to $q_{\text{cr}}$ for the shell under consideration with an infinite increase in ring stiffness. For a shell of "medium" length, $\underline{n}$ is equal to the integer part of the number $2.77(1-\nu^2)^{1/8}(m\rho)^{-1/2}(R/h)^{1/4}$. If the span of the shell between two neighboring rings is, as is usually the case, a shell of "medium" length, then $\overline{n}$ is equal to the integer part of the number $2.77(1-\nu^2)^{1/8}\rho^{-1/2}(R/h)^{1/4}$.

It is not difficult to establish by mathematical analysis that the graph of the quantity $qR/Eh$, determined by formula (8) (or (7′), if $m = 2$), for sufficiently small $\mu > 0$ lies in the positive half-plane above the graph of the quantity $qR/Eh$ determined by formula (6), and actually intersects it, passing into the negative half-plane when

$\mu < \bar{\mu}$. It can be shown that, with an infinite increase in the stiffness of the rings, the quantity $q_{\text{cr}}$, determined by the method described, will tend to $q_{\text{cr}}$ for a shell span with hinged edges, while with an infinite decrease in the stiffness of the rings (when $h \to \mathrm{H}$, $a \to 0$) it will tend to $q_{\text{cr}}$ for a shell with vanishing rings.* Hence it is clear that the relative error (in the direction of underestimating $q_{\text{cr}}$) obtained by ignoring the equality of the angles of rotation of the joined elements (see p. 549) cannot exceed the relative change in $q_{\text{cr}}$ in passing from hinged edge support to clamping. For a shell of "medium" length, according to the linear theory, $q_{\text{cr}}$ with clamped shell edges is 1.5 times greater than $q_{\text{cr}}$ with hinged edge support. This result is obtained if, in both cases, the precritical stressed state is taken to be momentless. Allowance for moments in the precritical state of a shell clamped along the edges, as shown by examples in ($^2$), reduces the indicated number 1.5 to 1.2-1.35. For a sufficiently long shell this number should practically be replaced by unity. Therefore the value of $q_{\text{cr}}$ found on the basis of the theory presented may be lower than the actual value of $q_{\text{cr}}$ for clamped shell edges by no more than about a factor of 1.4.

If the shell is infinitely long, then, putting $\alpha_i = 0$ $(i = 1, 2, \dots, m)$, we obtain, instead of equations (7), a single equation

$$\frac{qR}{Eh} = \frac{\mathrm{H}^3(n^2-1)}{12khR^2} - \mu\left[\frac{h^2(2-\nu-n^{-2})}{6(1-\nu^2)kaR} + \frac{2R}{ka}\frac{\mu^2}{n^4(n^2-1)}\right]\operatorname{tg}\frac{\mu\rho}{2}. \qquad (10)$$

From (6) and (10), the quantity $q_{\text{cr}}$ is determined by the indicated graphical method; here $0 < \mu < \pi/\rho$, $2 \le n \le \bar{n}$.

The experimental values $q_{\text{cr}}$ ($q_{\text{cr}}^{\text{e}}$) obtained by the authors agree quite well with the corresponding theoretical values $q_{\text{cr}}$ ($q_{\text{cr}}^{\text{t}}$), calculated on the basis of the theory presented. The tested shells, turned with great precision together with the reinforcing rings from steel tubes, were clamped along the edges and were shells of "medium" length. Therefore $q_{\mathrm{cr}}^{\mathrm{e}}$ corresponds to $1.4q_{\mathrm{cr}}^{\mathrm{t}}$. As an example, we give the following results. For a shell with 8 spans ($m = 8$), with $\rho = 0.347$, $R = 10.3$ cm, $h = 0.038$ cm, $\mathrm{H} = 0.119$ cm, $a = 0.2$ cm, we have $q_{\mathrm{cr}}^{\mathrm{t}} = 1.23$, $1.4q_{\mathrm{cr}}^{\mathrm{t}} = 1.72$, $q_{\mathrm{cr}}^{\mathrm{e}} = 1.72$ (kg/cm$^2$). If this shell were deprived of reinforcing rings, then $q_{\mathrm{cr}}^{\mathrm{t}} = 0.53$ kg/cm$^2$. If the number of spans of the indicated shell with rings is increased to infinity, then $q_{\mathrm{cr}}^{\mathrm{t}} = 0.066$ kg/cm$^2$, whereas for the corresponding infinitely long shell without rings $q_{\mathrm{cr}}^{\mathrm{t}} = 0.027$ kg/cm$^2$.

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## CITED LITERATURE

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* Then the discontinuity of the quantity $(w_i)'_\xi$ in passing from one span to another vanishes in the limit. For example, in the case $m = 2$ we have $\alpha_2 = -\alpha_1$ ($\alpha_1 = \mu\rho/2 - \pi/2$), $C_1 = C_2 = C$ (on the basis of (5)) and

$$w_1 = C\cos(\mu\xi + \mu\rho/2 - \pi/2)\sin n\varphi, \qquad w_2 = C\cos(\mu\xi - \mu\rho/2 + \pi/2)\sin n\varphi,$$

whence

$$(w_1)'_\xi\big|_{\xi=\rho/2} = -C\mu\sin(\mu\rho - \pi/2)\sin n\varphi, \qquad (w_2)'_\xi\big|_{\xi=-\rho/2} = C\mu\sin(\mu\rho - \pi/2)\sin n\varphi.$$

When $a \to 0$, $\mathrm{H} \to h$, then $\mu\rho \to \pi/2$,

$$(w_1)'_\xi\big|_{\xi=\rho/2} \to 0, \qquad (w_2)'_\xi\big|_{\xi=-\rho/2} \to 0.$$

*Note: Figure translations are in progress. See original paper for figures.*

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