MATHEMATICS

Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan and A. B. Nersesyan

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Abstract

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EXPANSIONS IN SPECIAL BIORTHOGONAL SYSTEMS AND BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

In this note we study special biorthogonal systems formed from linear combinations of functions of Mittag-Leffler type

$$E_\rho(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})} \qquad (\mu > 0,\ \rho \geq 1/2), \tag{1}$$

and state theorems on the equiconvergence of expansions in these systems with ordinary Fourier series. Next, mutually adjoint boundary-value problems are formulated for certain classes of differential equations of fractional order $1/\rho$ ($\rho \geq 1/2$) on a finite interval, whose eigenfunctions and associated functions coincide with the constructed biorthogonal systems for particular values of the parameters.

The results presented below are discrete analogues of the theory of special integral transformations developed in the work ([1]) of one of the authors of this note.

1°. **Construction of a biorthogonal system.** Introduce for consideration the functions

$$y(x,\lambda,\rho) = \sum_{j=1}^{2} a_j x^{\mu_j - 1} E_\rho(\lambda x^{1/\rho};\mu_j); \quad z(x,\lambda,\rho) = \sum_{j=1}^{2} b_j (l-x)^{\nu_j - 1} E_\rho(\lambda (l-x)^{1/\rho};\nu_j),$$

$$\omega(\lambda,\rho) = \lambda \sum_{j_1,j_2=1}^{2} a_{j_1} b_{j_2} l^{\mu_{j_1}+\nu_{j_2}-1} E_\rho(\lambda l^{1/\rho};\mu_{j_1}+\nu_{j_2}); \tag{2}$$

where $x \in (0,l)$; λ is a complex variable. The functions (2) are entire, have order ρ, and their type is $\leq l$.

For definiteness we shall assume that

$$\rho \geq 1/2, \qquad \mu_1 > \mu_2 \geq 0, \qquad \nu_1 > \nu_2 \geq 0, \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1. \quad (3)$$

It is proved that the integral formula

$$\int_0^l y(x, \lambda, \rho) z(x, \lambda^*, \rho)\, dx = \frac{\omega(\lambda, \rho) - \omega(\lambda^*, \rho)}{\lambda - \lambda^*} \quad (4)$$

holds for arbitrary λ and λ^*.

Under the assumption that

$$\tau_\rho = \max_{e_0}\{1 + \rho(1 - \mu_{j_1} - \nu_{j_2})\} = 1 + \rho(1 - \mu_r - \nu_s) \geq 0, \quad (5)$$

where e_0 is the set of those combinations of indices $j_1, j_2 (= 1, 2)$ for which $a_{j_1} b_{j_2} \neq 0$, the following propositions are established:

a) For any $A_0 \in (-\infty, +\infty)$, the function $\omega(\lambda, \rho) - A_0$ has a countable set of zeros (except, possibly, in the case $\rho = 1,\ \tau_1 = 0$, when the exceptional value $A_0 = -\operatorname{sign}(a_2 b_2)$ is possible).

For $\rho \geqslant 1/2$, all sufficiently large zeros in modulus of the function $\omega(\lambda, \rho) - A_0$ are simple (with the possible exception of the case $\rho = 1/2,\ \tau_{1/2} = 0$, when $1 + A_0 \operatorname{sign}(a_2 b_2) = +1$).

b) If $\rho = 1/2$, then all zeros with sufficiently large modulus lie on the half-axis $(-\infty, 0)$, and when they are numbered in the order of nonincreasing moduli, the asymptotic formula holds

$$\lambda_k = -\left(\frac{\pi k}{l}\right)^2 \left\{1 + O\left(\frac{1}{k}\right)\right\}. \quad (6)$$

If, however, $\rho > 1/2$, then, when numbering the zeros lying respectively in the half-planes $\operatorname{Im} \lambda > 0$ and $\operatorname{Im} \lambda < 0$, the asymptotic formulas hold

$$\lambda_k^{(\pm)} = e^{\pm i\pi/\rho} \left(\frac{2\pi k}{l}\right)^{1/\rho} \left\{1 + O\left(\frac{\log k}{k}\right)\right\}. \quad (7)$$

Let $\{\lambda_k\}$ $(k = 1, 2, ...)$ be the sequence of all zeros of the function $\omega(\lambda, \rho) - A_0$, numbered in the order of nonincreasing moduli, independently of their multiplicities.

c) With a zero λ_n of multiplicity $p_n \geqslant 1$, associate two systems of functions

$$\left.\frac{\partial^j y(x,\lambda,\rho)}{\partial\lambda^j}\right|_{\lambda=\lambda_n}, \qquad \left.\sum_{k=0}^{p_n-j-1}\frac{b^{(n)}_{p_n-j-k-1}}{k!\,j!}\,\frac{\partial^k z(x,\lambda,\rho)}{\partial\lambda^k}\right|_{\lambda=\lambda_n} \qquad (j=0,1,\dots,p_n-1), \tag{8}$$

where

$$b_k^{(n)} = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left.\left\{\frac{(\lambda-\lambda_n)^{p_n}}{\omega(\lambda,\rho)-A_0}\right\}\right|_{\lambda=\lambda_n} \qquad (k=0,1,\dots,p_n-1). \tag{9}$$

Number all groups* of functions (8) in the order of nonincreasing numbers $|\lambda_n|$, and in such a way that, for each λ_n $(n=1,2,\dots)$, the corresponding j-th functions from these groups are assigned identical numbers. It follows from (4) that the two countable systems of functions thus obtained, $\{Y_k(x,\rho)\}$ and $\{Z_k(x,\rho)\}$ $(k=1,2,\dots)$, are biorthogonal on the interval $[0,l]$ in the sense

$$\int_0^l Y_p(x,\rho)Z_q(x,\rho)\,dx = \begin{cases}0, & p\neq q,\\ 1, & p=q\end{cases} \qquad (p,q=1,2,\dots). \tag{10}$$

2°. Expansion theorems.

Assign to the class $L_1\{0,l\}$ all functions from the class $L_1(0,l)$ that are bounded in certain neighborhoods of the points $x=0$ and $x=l$.

Theorem 1. Let $\rho=1/2$, and let the parameters μ_{j_1},ν_{j_2} $(j_1,j_2=1,2)$ satisfy conditions (3), (5), as well as the condition

$$\max(\mu_r,\nu_s)<3. \tag{11}$$

If $f(x)\in L_1\{0,l\}$, then uniformly with respect to $x\in[\varepsilon,l-\varepsilon]$ (where $0<\varepsilon<l/2$ is arbitrary) the equality

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\int_0^l f(t)Z_k(t,1/2)\,dt\right)Y_k(x,1/2)-\frac{1}{\pi}\int_0^l f(t)\frac{\sin\frac{\pi n}{l}(x-t)}{x-t}\,dt\right\}=0. \tag{12}$$

An analogous assertion holds for the expansion of a function $f(x)\in L_1\{0,l\}$ in the system $\{Z_k(x,1/2)\}$ $(k=1,2,\dots)$.

* We note that, by a), $p_n=1$ for $n\gg N$, except for the case indicated there, when $p_n=2,\ n\gg 2$.

To formulate the expansion theorem in the case $\rho > 1/2$, some definitions and notation are needed. Let $f(x) \in L_1(0,l)$; then the **fractional integral of order** $\alpha > 0$ **with origin at the point** $x = 0$ is the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \in L_1(0,l), \tag{13}$$

and the **fractional integral of order** $\alpha > 0$ **with endpoint at the point** $x = l$ is the function

$$\frac{d^{-\alpha}}{d(l-x)^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^l (t-x)^{\alpha-1} f(t)\, dt \in L_1(0,l). \tag{14}$$

In the case $\alpha = 0$, naturally, both fractional integrals (13) and (14) are to be identified with the function $f(x)$ itself.

We shall agree to say that $f(x) \in L_1^{(\alpha)}[0,l]$ $(\alpha > 0)$ if $f(x)$ is a fractional integral of the form (13) of some function $g_f(x) \in L_1\{0;l\}$.

Similarly, we shall say that $f(x) \in L_1^{(\alpha)}\{0,l\}$ $(\alpha > 0)$ if it is a fractional integral of the form (14) of some function $g_f(x) \in L_1\{0;l\}$.

Then it is natural to regard the classes $L_1^{(0)}[0,l]$, $L_1^{(0)}\{0,l\}$, and $L_1\{0,l\}$ as identical. We further denote

$$A_k^{(\rho)} = -\sum_{j_1,j_2=1}^{2} \frac{a_{j_1} b_{j_2} l^{\mu_{j_1}+\nu_{j_2}-k\rho^{-1}-1}}{\Gamma(\mu_{j_1}+\nu_{j_2}-k\rho^{-1})} \qquad (k = 1, 2, \ldots) \tag{15}$$

and, assuming that $A_2^{(\rho)} \neq 0$, introduce the functions

$$Y_0(x,\rho) = \sum_{j=1}^{2} \frac{a_j x^{\mu_j-\rho^{-1}-1}}{\Gamma(\mu_j-\rho^{-1})}, \qquad Z_0(x,\rho) = -\frac{1}{A_2^{(\rho)}} \sum_{j=1}^{2} \frac{b_j (l-x)^{\nu_j-\rho^{-1}-1}}{\Gamma(\nu_j-\rho^{-1})}. \tag{16}$$

For $A_0 = A_1^{(\rho)}$, we additionally set

$$\rho > 1, \qquad A_2^{(\rho)} \neq 0, \qquad \min\{\mu_r, \nu_s\} > \frac{1}{\rho}. \tag{17}$$

In this case the systems $\{Y_p(x,\rho)\}$ and $\{Z_p(x,\rho)\}$ $(p = 0, 1, 2, \ldots)$ are biorthogonal on $[0,l]$ in the sense of formula (10). We shall assume that $\alpha = \tau_p/\rho$ when $A_0 \neq A_1^{(\rho)}$ and $\alpha = (\tau_p + 1)/\rho$ when $A_0 = A_1^{(\rho)}$.

Theorem 2. If $\mu_r > 0$ and $f(x) \in L_1^{(\alpha)}[0,l]$, then, uniformly with respect to $x \in [\varepsilon, l-\varepsilon]$ (where $0 < \varepsilon < l/2$ is arbitrary), the limiting equality

$$\begin{gathered}\lim_{n\to\infty}\left\{\sum_{k=1}^{n} Y_k(x,\rho)\int_0^l f(t)Z_k(t,\rho)\,dt - \frac{1}{\pi}\int_0^l f(t)\frac{\sin\frac{\pi n}{l}(x-t)}{x-t}\,dt\right\} \\ = \begin{cases} 0, & \text{when } A_0 \neq A_1^{(\rho)}, \\ -\left(1-\frac{1}{2\rho}\right)Y_0(x,\rho)\int_0^l f(t)Z_0(t,\rho)\,dt, & \text{when } A_0 = A_1^{(\rho)}. \end{cases}\end{gathered} \tag{18}$$

An analogous theorem also holds for the expansion of a function $f(x) \in L_1^{(\alpha)}\{0,l\}$ in the system $\{Z_k(x,\rho)\}$ $(k = 1,2,...)$, provided $\nu_s > 0$.

Let us note a somewhat unusual fact, consisting in the presence of the factor $(1-\frac{1}{2\rho})$ in formula (18) when $A_0 = A_1^{(\rho)}$. Although we do not yet have an exhaustive explanation of this fact, we nevertheless note that it does not contradict the biorthogonality of the systems $\{Y_k(x,\rho)\}$ and $\{Z_k(x,\rho)\}$ $(k = 0,1,2,...)$, since the function $Y_0(x,\rho)$, for example, does not belong to the class $L_1^{(\alpha)}[0,l]$.

3°. **Boundary-value problem.** As is known, a function $g(x) \in L_1(0,l)$ is called the fractional derivative of a function $f(x) \in L_1(0,l)$ of order $\alpha > 0$ with origin at the point $x = 0$, if $f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}}g(x)$. Denoting then

$$g(x) = \frac{d^\alpha}{dx^\alpha}f(x) \tag{19}$$

in what follows, by the symbol d^α/dx^α we shall mean the fractional integral for $\alpha \leqslant 0$ and the fractional derivative for $\alpha > 0$. The fractional derivative $d^\alpha/d(l-x)^\alpha$ of a function $f(x) \in L_1(0,l)$ of order $\alpha > 0$ with terminal point at $x = l$ is defined analogously.

Let the real parameters ξ, η and A_0 be arbitrary, while $\alpha_1, \alpha_2, \alpha_3$ and ρ are connected by the relation $1/\rho = 2 + \alpha_1 + \alpha_2 + \alpha_3$ and satisfy the conditions $\rho \geqslant 1/2,\ -1 < \alpha_1, \alpha_3 < 1/\rho,\ \alpha_2 > -2$.

Consider in the class of functions $L_1(0,l)$ the following boundary-value problems, which it is natural to call mutually adjoint:

$$(A)\begin{cases} \dfrac{d^{\alpha_3}}{dx^{\alpha_3}}\dfrac{d}{dx}\dfrac{d^{\alpha_2}}{dx^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_1}}{dx^{\alpha_1}}y(x) = \lambda y(x) \qquad (0 \leqslant x \leqslant l), \\ \dfrac{d^{\alpha_1}}{dx^{\alpha_1}}y(x)\Big|_{x=0} = \sin\xi, \qquad \dfrac{d^{\alpha_2}}{dx^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_1}}{dx^{\alpha_1}}y(x)\Big|_{x=0} = -\cos\xi, \\ \dfrac{d^{\alpha_1}}{dx^{\alpha_1}}y(x)\Big|_{x=l}\cos\eta + \dfrac{d^{\alpha_2}}{dx^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_1}}{dx^{\alpha_1}}y(x)\Big|_{x=l}\sin\eta = A_0; \end{cases} \tag{20-22}$$

$$
(A^*)\begin{cases}
\dfrac{d^{\alpha_1}}{d(l-x)^{\alpha_1}}\dfrac{d}{dx}\dfrac{d^{\alpha_2}}{d(l-x)^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_3}}{d(l-x)^{\alpha_3}}z(x)=\lambda z(x) \qquad (0\leqslant x\leqslant l),\\
\dfrac{d^{\alpha_3}}{d(l-x)^{\alpha_3}}z(x)\Big|_{x=0}\cos\xi+\dfrac{d^{\alpha_2}}{d(l-x)^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_3}}{d(l-x)^{\alpha_3}}z(x)\Big|_{x=0}\sin\xi=A_0,\\
\dfrac{d^{\alpha_3}}{d(l-x)^{\alpha_3}}z(x)\Big|_{x=l}=\sin\eta, \qquad \dfrac{d^{\alpha_2}}{d(l-x)^{\alpha_2}}\dfrac{d}{dx}\dfrac{d^{\alpha_3}}{d(l-x)^{\alpha_3}}z(x)\Big|_{x=l}=-\cos\eta.
\end{cases} \tag{20*-22*}
$$

Denote by $y(x,\lambda)$ the solution of problem (20), (21), and by $z(x,\lambda)$ the solution of problem $(20^*),(22^*)$. The solutions of these problems in the class $L_1(0,l)$ exist and are unique. If in formulas (2) we put

$$
\mu_1=1+\alpha_1, \qquad \mu_2=2+\alpha_1+\alpha_2, \qquad \nu_1=1+\alpha_3, \qquad \nu_2=2+\alpha_2+\alpha_3,
$$
$$
a_1=\sin\xi, \qquad a_2=-\cos\xi, \qquad b_1=\sin\eta, \qquad b_2=\cos\eta, \tag{23}
$$

then it turns out that

$$
y(x,\lambda,\rho)\equiv y(x,\lambda), \qquad z(x,\lambda,\rho)\equiv z(x,\lambda) \qquad (0\leqslant x\leqslant l), \tag{24}
$$

and the eigenvalues of problems (A) and (A^*) coincide and are the A_0-points of the function $\omega(\lambda)\equiv\omega(\lambda,\rho)$. Thus, for the special parameter values (23), the biorthogonal system (10) is a system of eigenfunctions and associated functions of the problems (A) and (A^*), and Theorems 1 and 2 are theorems on the equiconvergence of expansions in the systems of eigenfunctions and associated functions of the problems (A) and (A^*) with ordinary Fourier series.

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

Yerevan State University

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1. M. M. Dzhrbashyan, *Izv. AN SSSR, Ser. Matem.*, **19**, 133 (1955).

Note: Figure translations are in progress. See original paper for figures.