Soviet-era science, translated into English

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1960

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Abstract

Full Text

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ON THE THEORY OF DIFFERENTIAL EQUATIONS $\dfrac{dx}{dt} = f(x,t)$ IN LOCALLY CONVEX SPACES

(Presented by Academician S. L. Sobolev on 23 XI 1959)

In this paper, some theorems known for the equation $dx/dt = f(x,t)$ in Banach spaces (^{1,12}) are generalized to the case of complete locally convex spaces.

Let E be a complete locally convex space and let $\{p(x)\}$ be a sufficient set of seminorms in E (^{2}). Let $\mathfrak{F}(S,E)$ be the set of mappings of a measurable set $S \subseteq E^n$ ($\operatorname{mes} S < \infty$) into E (E^n is an n-dimensional Euclidean space).

A function $x(\alpha) \in \mathfrak{F}(S,E)$ is called **integrable on** S if there exists a directed sequence of functions $x_\beta(\alpha) \in \mathfrak{F}(S,E)$ (where $\beta \in B$; B is a directed set) such that:

1) each $x_\beta(\alpha)$ is a countably-valued function (^{3});
2) for each seminorm $p(x) \in \{p(x)\}$,

$$I_S(x_\beta(\alpha)) = \sum_{i=1}^{\infty} p[x_\beta(\alpha_{i\beta})] \operatorname{mes} S_{i,\beta} < \infty,$$

   where $S_{i,\beta}$ ($i = 1, \dots, n, \dots$) are sets on which $x_\beta(\alpha)$ is constant, and $\alpha_{i\beta} \in S_{i,\beta}$;
3) for every $\varepsilon > 0$ and $p(x) \in \{p(x)\}$ there exists $\beta \in B$ such that

$$I_S(x_{\beta'}(\alpha) - x_{\beta''}(\alpha)) < \varepsilon$$

   for all $\beta', \beta'' > \beta$; $\beta', \beta'' \in B$;
4) for every $\varepsilon > 0$ there exists $S_\varepsilon \subset S$, $\operatorname{mes} S_\varepsilon < \varepsilon$, such that $\{x_\beta(\alpha)\}$ converges uniformly on $S \setminus S_\varepsilon$ to $x(\alpha)$.

Then

$$I = \lim_{\beta \in B} \sum_{i=1}^{\infty} x_\beta(\alpha_{i\beta}) \operatorname{mes} S_{i,\beta}$$

exists and does not depend on the choice of $\{x_\beta(\alpha)\}$ with properties 1)–4).

We call I the **integral**

$$\int_S x(\alpha)\,d\alpha$$

(a more general integral was introduced in (^{4})).

The integral introduced has the following properties:

1°. I is a completely additive set function.

2°. I is an absolutely continuous set function.

3°. If $x(\alpha)$ and $y(\alpha)$ are integrable, then

$$\int_S [\mu x(\alpha) + \nu y(\alpha)]\,d\alpha = \mu \int_S x(\alpha)\,d\alpha + \nu \int_S y(\alpha)\,d\alpha.$$

4°. If $x(\alpha)$ is integrable, then $p(x(\alpha))$ is summable and

$$p\left(\int_S x(\alpha)\,d\alpha\right) \le \int_S p(x(\alpha))\,d\alpha.$$

5°. Let $x(\alpha)$ be defined on a compact set $S \subset E^n$, and let the set $\mathfrak{X}$ of its values be bounded (^{5}). Suppose that for every $\varepsilon > 0$ there exists $S_\varepsilon \subset S$,

$\operatorname{mes} S_\varepsilon < \varepsilon$, such that $x(\alpha)$ is continuous on $S \setminus S_\varepsilon$. Then $x(\alpha)$ is integrable on S and $\int_S x(\alpha)\,d\alpha \in \operatorname{mes} S \cdot [\mathfrak{X}]$, where $[\mathfrak{X}]$ is the closed convex hull of the set $\mathfrak{X}$.

6°. If $x(\alpha)$ is integrable on S and continuous at $\alpha_0 \in S$ (with respect to S), then

$$\lim_{d(Q)\to 0,\ \alpha_0 \in Q} \frac{1}{\operatorname{mes} Q} \int_Q x(\alpha)\,d\alpha$$

exists and is equal to $x(\alpha_0)$.

7°. The system

$$x(t_0) = x_0, \qquad \frac{dx}{dt} = f(x,t), \qquad x(t) \in M \subseteq E \quad \text{for } |t - t_0| \le a,$$

where $f(x,t)$ maps $M \times E^1$ continuously into E, is equivalent to the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau),\tau)\,d\tau.$$

Definition. An operator A ($A(E) \subseteq E$) is called **contracting** if there exists $0 \le q < 1$ such that for all $p(x) \in \{p(x)\}$, $x, y \in E$,

$$p(A(x) - A(y)) \le q p(x - y). \tag{1}$$

Theorem 1. Let A be a contracting operator, $\emptyset \ne T = \overline{T} \subseteq E$, $A(T) \subseteq T$.

Then there exists, and moreover is unique in E, an $x \in T$ such that $A(x) = x$.

Theorem 2 (^{6}). Let T be a closed convex set $\subseteq E$. Suppose operators A_i $(A_i(E) \subseteq E,\ i = 1, 2)$ are given on T, with: 1) A_1 a contracting operator; 2) $A_2(T)$ bicompact; 3) A_2 continuous; 4) if $x, y \in T$, then $A_1(x) + A_2(y) \in T$.

Then there exists $x \in T$ such that $A_1(x) + A_2(x) = x$.

Proof. Let $x \in T$. Then (conditions 1), 4) and Theorem 1) there exists, and moreover is unique, $y \in T$ such that $y = A_1(y) + A_2(x)$, i.e., an operator $y = C(x)$ is defined on T, for which

$$C(x) = A_1 C(x) + A_2(x), \qquad C(T) \subseteq T. \tag{2}$$

Let $z, v \in T$. Then from (2) and condition 1) we obtain

$$p(C(z) - C(v)) \leq \frac{1}{1-q} p(A_2(z) - A_2(v))$$

for every seminorm $p(x) \in \{p(x)\}$. Hence, by conditions 2), 3), C is continuous and $\overline{C(T)}$ is bicompact (the latter with the aid of (^{8})). Considering C only on the closed convex hull of the set $\overline{C(T)}$ (^{7}) and applying Tikhonov' s principle (^{9}), we obtain the assertion of the theorem.

Let $\widetilde{E}_S$ be the space of uniform convergence on compacta of continuous mappings of the set $S \subseteq E^1$ into E. Then $\widetilde{E}_S$ is a complete locally convex space with a sufficient set of seminorms

$$\{p_{p,B}(\tilde{x}) = \sup_{t \in B} p(x(t))\}, \qquad [\tilde{x} = x(t) \in \widetilde{E}_S],$$

where $p(x)$ ranges over $\{p(x)\}$, and B ranges over some covering of S by compacta. $\widetilde{M}_S$ denotes the set of mappings of S into $M \subseteq E$.

Lemma 1. Let $f(x, t)$ map $M \times S$ continuously into E, where $M \subseteq E$, $S = [t_0 - h, t_0 + h] \subset E^1$. Let $K(t) \geq 0$ be a real function such that

$$(L) \left| \int_{t_0}^{t} K(\tau)\, d\tau \right| \leq q < 1 \quad (t \in S)$$

and for all $x, y \in M$, $p(x) \in \{p(x)\}$,

$$p(f(x,t) - f(y,t)) \leq K(t) p(x - y).$$

Then

$$A(\tilde{x}) = x_0 + \int_{t_0}^{t} f(x(\tau), \tau)\, d\tau$$

is a contraction operator defined on $\tilde{M}_S$.

The lemma follows from properties $5^\circ, 2^\circ, 4^\circ$ of the integral.

Theorem 3. Let $f(x,t)$ continuously map $U \times [t_0 - a, t_0 + a]$ into E, where

$$U = U(x : p_i(x - x_0) \le \varepsilon), \quad p_i(x) \in \{p(x)\} \quad (i = 1, \dots, n),$$

and suppose

$$\sup_{x \in U;\ |t - t_0| \le a} p_i(f(x,t)) < \infty \quad (i = 1, \dots, n).$$

Let $K(t) \ge 0$ be summable on $[t_0 - a, t_0 + a]$, and for all $x, y \in U$, $p(x) \in \{p(x)\}$,

$$p(f(x,t) - f(y,t)) \le K(t) p(x - y).$$

Then there exists h_0, $0 < h_0 \le a$, such that for every h, $0 < h \le h_0$, there exists, and moreover is unique, a solution $x(t)$ of the initial-value problem

$$dx/dt = f(x,t), \qquad x(t_0) = x_0,$$

defined on $[t_0 - h, t_0 + h]$.

Proof. Put $h_0 = \min(h_1, h_2)$, where h_1 is such that

$$\max\left(\int_{t_0}^{t_0 + h_1} K(\tau)\, d\tau,\ \int_{t_0 - h_1}^{t_0} K(\tau)\, d\tau \right) \le q < 1, \qquad h_2 = \frac{\varepsilon}{\sup\limits_{\substack{x \in U;\ |t - t_0| \le a \\ i = 1, \dots, n}} p_i(f(x,t))}.$$

Let $h < h_0$. Then, by Lemma 1, the choice $h_2 > h$, and Theorem 1, the operator

$$A(\tilde{x}) = x_0 + \int_{t_0}^{t} f(x(\tau), \tau)\, d\tau$$

has a unique fixed point in the closed set

$$\tilde{U}_{[t_0 - h, t_0 + h]} \subset \tilde{E}_{[t_0 - h, t_0 + h]}.$$

Since for every solution $x(t)$ of the system $dx/dt = f(x,t)$, $x(t_0) = x_0$, there exists $\delta > 0$ such that $x(t) \in U$ for $|t - t_0| < \delta$, the theorem is proved.

Remark. If in the hypothesis of Theorem 3 one replaces U by all of E and

$$\int_{-\infty}^{+\infty} K(\tau)\, d\tau < 1,$$

then there exists a solution defined on the entire line.

Lemma 2. Let $f(x,t)$ continuously map $M \times S$ into E, where $M \subseteq E$, $S \subseteq E^1$.

Then the operator $F(\tilde{x}) = f(x(t), t)$: 1) is defined on $\tilde{M}_S$ and takes values in $\tilde{E}_S$; 2) is continuous.

Proof. 1) See (10). The idea of the proof of 2) is that a continuous operator $f(x, t)$ is uniformly continuous with respect to each bicompact set $\mathfrak{X} \times B$, where $B \subseteq S$ is an arbitrary compact set, and $\mathfrak{X}$ is the set of values on it of an arbitrary $\tilde{x} \in \tilde{E}_S$.

Lemma 3. Let $f(x, t)$ continuously map $M \times S$ into E ($M \subseteq E$, $S \subseteq E^1$), and suppose that for each compact $B \subseteq S$ there exists a bicompact $F_B \subseteq E$ such that

$$f(M \times B) \subseteq F_B.$$

Then the operator

$$A(\tilde{x}) = \int_{t_0}^{t} f(x(\tau), \tau)\, d\tau :$$

1) is continuous on $\tilde{M}_S$ and takes values in $\tilde{E}_S$; 2) $A(\tilde{M}_S)$ is bicompact.

Proof. 1) follows from Lemma 2 and property 4° of the integral. 2) follows from property 5° of the integral and Ascoli' s theorem (11).

Theorem 4. Let $f(x, t) = f_1(x, t) + f_2(x, t)$, where $f_1(x, t)$, $f_2(x, t)$ continuously map $U \times [t_0 - a, t_0 + a]$ into E

$$(U = U(x : p_i(x - x_0) \leq \varepsilon,\ p_i(x) \in \{p(x)\},\ i = 1, \dots, n)).$$

Suppose $\overline{f_2(U \times [t_0 - a, t_0 + a])}$ is bicompact, and $f_1(x, t)$ satisfies the hypotheses of Theorem 3.

Then there exists $h > 0$ such that there is a solution of the initial-value problem

$$x(t_0) = x_0, \qquad dx/dt = f(x, t),$$

defined on $[t_0 - a, t_0 + a]$.

Theorem 5$^{(12)}$. Let $f(x, t)$ satisfy the conditions of Lemma 3, where $M = E$, $S = [t_0, +\infty)$, and let there exist $p_0(x) \in \{p(x)\}$ and a continuous function $G(r, t)$, nondecreasing in r ($t \geqslant 0,\ r \geqslant 0$), such that for $x \in E,\ t \in S$

$$p_0(f(x, t)) \leqslant G(p_0(x), t). \tag{3}$$

Suppose that for every $r_0 > 0$ there exists a function $g(t)$, defined on S, such that

$$\frac{dg}{dt} \geqslant G(g(t), t), \qquad g(t_0) = r_0. \tag{4}$$

Then for every $x_0 \in E$ there exists a solution of the initial-value problem

$$dx/dt = f(x,t), \qquad x(t_0) = x_0,$$

defined on all of S.

Proof. On the closed convex set

$$T = T(\tilde{x} = x(t):\ p_0(x(t)) \leqslant g(t)) \quad (T \subset \widetilde{E}_S)$$

define the operator

$$A(\tilde{x}) = x_0 + \int_{t_0}^{t} f(x(\tau),\tau)\,d\tau.$$

By Lemma 3, $\overline{A(T)}$ is bicompact $\subset \widetilde{E}_S$, and A is continuous on T. From (3) and (4) we infer $A(T) \subseteq T$. Applying Tikhonov' s principle, we complete the proof.

Theorems 6 and $7^{(12)}$ are proved analogously.

Theorem 6. Let the conditions of Theorem 5 be fulfilled for every $p_0(x) \in \{p(x)\}$, and let $g(t)$ be bounded.

Then the solution $x(t)$ of the initial-value problem is bounded (i.e. the set of values of the function $x(t)$ is bounded$^{(5)}$).

Theorem 7. Let the conditions of Theorem 5 be fulfilled for every $p_0(x) \in \{p(x)\}$, where (3) may be fulfilled only for x in some neighborhood of zero, and in (4) there is strict equality. Let $f(0,t) = 0$, $G(0,t) = 0$, and let the point $g = 0$ for the equation $dg/dt = G(g(t),t)$ be stable (asymptotically stable).

Then $x = 0$ is a stable (respectively, asymptotically stable) point for the equation $dx/dt = f(x,t)$.

I express my gratitude to V. V. Nemytskii for posing the problem and for his guidance.

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Received
20 XI 1959

CITED LITERATURE

[1] M. A. Krasnosel' skii, S. G. Krein, *Proceedings of the Seminar on Functional Analysis*, Voronezh State Univ., no. 2 (1956).
[2] M. A. Naimark, *Normed Rings*, Moscow, 1956, pp. 46–52.
[3] E. Hille, *Functional Analysis and Semigroups*, IL, 1951, p. 54, Definition 3.2.2 (3).
[4] R. S. Phillips, Trans. Am. Math. Soc., **47**, no. 1, 114 (1940).

[5] I. M. Gel’ fand, G. E. Shilov, *Generalized Functions*, vol. 2, *Spaces of Basic and Generalized Functions*, Moscow, 1958, p. 45.
[6] M. A. Krasnosel’ skii, *Uspekhi Mat. Nauk*, **10**, no. 1 (63), 123 (1955).
[7] N. Bourbaki, *Vector Topological Spaces*, IL, 1959, Ch. II, § 4, item 1.
[8] N. Bourbaki, *General Topology*, Moscow, 1958, Ch. II, § 4, Theorem 3.
[9] A. Tychonoff, Math. Ann., **111**, 767 (1935).
[10] N. Bourbaki, *General Topology*, Moscow, 1958, Ch. I, § 4, Theorem 1.
[11] N. Bourbaki, *Topologie générale*, 1953, Ch. X, § 4, théorème 1.
[12] A. Stokes, Proc. Nat. Acad. Sci. USA, **45**, no. 2, 231 (1959).

Note: Figure translations are in progress. See original paper for figures.

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