MATHEMATICS

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1960

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Abstract

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ON A CLASS OF HYPOELLIPTIC SYSTEMS

(Presented by Academician I. G. Petrovsky on 4 VI 1960)

1. A differential operator $\mathcal{P}$ is called **hypoelliptic** in a domain I if, for every generalized function u, infinite differentiability of $\mathcal{P}u$ in I implies infinite differentiability of u in every subdomain $I' \subset I$.

The question of hypoellipticity of a **single** differential equation has by now been well studied. For equations with constant coefficients, Hörmander (1,2) indicated a necessary and sufficient condition for hypoellipticity, and asymptotically sharp estimates of derivatives were obtained for solutions of hypoelliptic equations. In the case of equations with variable infinitely differentiable coefficients, hypoellipticity was proved in (3,4) for a broad class of so-called "formally hypoelliptic" equations*. In contrast to the case of a single equation, hypoellipticity of systems of equations has been proved only in two special cases*∗: by Mizohata (5) for systems p-parabolic in the sense of Petrovsky, and by Douglis and Nirenberg (6) for elliptic systems. Broader classes of hypoelliptic systems are not known* ∗ ∗.

If one turns to the proofs of hypoellipticity of p-parabolic and elliptic systems, it is easy to notice that the passage from constant coefficients to variable ones causes no fundamental difficulties, because in these systems one can reasonably single out a principal part, and the local properties of solutions are determined mainly by this principal part.

In the present work a class of systems of equations will be singled out which we shall call quasi-elliptic (or q-quasi-elliptic). The class of quasi-elliptic systems will include elliptic systems (in the sense of (6)) and p-parabolic systems (in the sense of Petrovsky). In quasi-elliptic systems it is possible to single out a principal part and thereby, in the study of local properties, to eliminate the difficulties connected with the variability of the coefficients. Quasi-elliptic systems with infinitely differentiable coefficients will be hypoelliptic. Moreover, it can be shown that,

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[*] The operator $\mathcal{P}(x, D)$ is called **formally hypoelliptic** in the domain I if: a) for all $x' \in I$ the operators with constant coefficients $\mathcal{P}(x', D)$ are hypoelliptic; b) for any $x', x'' \in I$ the differential operators $\mathcal{P}(x'', D)$ and $\mathcal{P}(x', D)$ are equally strong in the sense of Hörmander (see (1), Chap. II).

$^{*}*$ Speaking of systems, we mean systems with variable coefficients, for the study of local properties of systems of equations with constant coefficients reduces to the study of local properties of a single equation.

$^{*}**$ Let us note that up to now there is no answer to the following question. In a domain I a system $\mathcal{P}(x, D)$ is given, and for all $x' \in I$ the operators with constant coefficients $\mathcal{P}(x', D)$ are hypoelliptic. What additional conditions must be imposed on the system so that it be hypoelliptic? The requirement that the operators with constant coefficients $\mathcal{P}(x', D)$ be equivalent is, in the case of systems, too strong and is not satisfied even for some systems elliptic in the sense of (^{6}).

that if the coefficients of a quasi-elliptic system belong to a certain Gevrey class, then all solutions will also belong to the same class. These results generalize the known theorems on the analyticity of solutions of elliptic systems (^{8}) and on the analyticity of solutions of p-parabolic systems with respect to the spatial variables (^{9}).

2. Consider a polynomial in ν variables $\xi = (\xi_1, \dots, \xi_\nu)$

$$P(\xi) = \sum_{\alpha \in \pi} a_\alpha \xi^\alpha = a_1 \xi_1^{m_1} + \dots + a_\nu \xi_\nu^{m_\nu} + \sum_{\alpha \in \pi'} a_\alpha \xi^\alpha *, \tag{1}$$

where π, π' are finite sets of integer vectors, and π' consists of those vectors for which either there exist two nonzero coordinates, or, if only the i-th coordinate α_i is nonzero, then $\alpha_i < m_i$. Put $m = \max m_i$, $q_i = m m_i^{-1}$ $(i = 1, \dots, \nu)$, $q = (q_1, \dots, q_\nu)$.

To each monomial ξ^α we shall assign the number $(\alpha, q) = \alpha_1 q_1 + \dots + \alpha_\nu q_\nu$, which will be called the order of ξ^α with respect to q. By $P_q(\xi)$ denote the sum of the monomials entering into the polynomial (1) whose order (with respect to q) is maximal. The polynomial $P_q(\xi)$ will be called the principal part of the polynomial $P(\xi)$ with respect to the vector q.

Definition 1. The polynomial (1) is called a **quasi-elliptic polynomial of weight** q if: a) $(\alpha, q) \le m$ for all vectors $\alpha \in \pi$; b) there exists a constant $h > 0$ such that for all real ξ

$$h^{-1} \sum_{i=1}^{\nu} |\xi_i|^{m_i} \le |P_q(\xi)| \le h \sum_{i=1}^{\nu} |\xi_i|^{m_i}.$$

We now consider a system of equations

$$\mathcal{P}(x, D) u = f, \tag{2}$$

where $u = (u_1, \dots, u_n)$, $f = (f_1, \dots, f_n)$;

$$\mathcal{P}(x, D) = \|\mathcal{P}_{ij}(x, D)\|_1^n, \qquad \mathcal{P}_{ij}(x, D) = \sum_{\alpha \in \pi_{ij}} a_{ij}^{(\alpha)}(x) D^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_\nu)$ is an integer vector; $D_k = \dfrac{1}{i}\dfrac{\partial}{\partial x_k}$, $x = (x_1, \dots, x_\nu)$.

Let $q = (q_1, \dots, q_\nu)$ $(q_i \geq 1,\ i = 1, \dots, \nu)$ be an arbitrary vector. As in (^{10}), associate with the system (2) the "order" matrix (with respect to q) $\mathfrak{M}(q) = \|m_{ij}(q)\|_1^n$, where $m_{ij}(q) = \max(\alpha, q)$, and the maximum is taken over all $\alpha \in \pi_{ij}$. To each substitution

$$T = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

associate the number $M_T = m_{1i_1}(q) + \dots + m_{ni_n}(q)$. The number $M(q)$, equal to the maximum of the numbers $M_T(q)$ over all substitutions T, will be called the order of the system with respect to the vector q.

Definition 2.** The system (2) will be called a **quasi-elliptic system of weight** q in the domain I if: a) at every point $x \in I$ the polynomial $P(x, \xi) = \det\|\mathcal{P}(x, \xi)\|$ is a quasi-elliptic polynomial of weight $q = (q_1, \dots, q_\nu)$, $q_1 = m/m_1, \dots, q_\nu = m/m_\nu$***, where*** m ***is the order of the polynomial*** $P(x, \xi)$***; b) the system (2) is nondegenerate with respect to the vector*** q***, i.e.*** $M(q) = m^*$.

* As usual, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_\nu^{\alpha_\nu}$.

**** Correction note.** In the recently published paper (^{12}), parabolic systems were introduced in which different weights are assigned to such different variables.

*** It can be shown that every quasi-elliptic polynomial has the same structure as the polynomial (1).

**** The meaning of condition b) is that, if $\det\|\mathcal{P}(x, \xi)\|$ is expanded by the usual rules,

the principal terms (with respect to q) do not cancel.

Using the results of paper (10), one can show that there exist $2n$ fractions $s_1, \dots, s_n, t_1, \dots, t_n$ (with denominator $\mu = m_1 \dots m_\nu$) such that

$$m_{ij}(q) \leqslant s_i + t_j, \qquad s_1 + \dots + s_n + t_1 + \dots + t_n = M(q) = m. \tag{3}$$

Put

$$\mathcal{P}_q(x, D) = \|\mathcal{P}_{q,ij}(x, D)\|_1^n; \qquad \mathcal{P}_{q,ij}(x, D) = \sum_{(\alpha,q)=s_i+t_j} a_{ij}^{(\alpha)}(x) D^\alpha. \tag{4}$$

If the numbers $s_1, \dots, t_n$ satisfy conditions (3), then the polynomial $\chi(x, \xi) = \det\|\mathcal{P}(x, \xi)\|$ will coincide with the principal part (relative to q) of the polynomial $P(x, \xi) = \det\|\mathcal{P}(x, \xi)\|$. The operator (4) is naturally called the principal part of operator (2). If system (2) is quasielliptic, then the polynomial $\chi(x, \xi) \neq 0$ for real $\xi \neq 0$, $x \in I$. The polynomial $\chi(x, \xi)$ is naturally called characteristic.

3. For quasielliptic systems two theorems hold:

Theorem 1. *If the right-hand side and the coefficients of system (2) are infinitely differentiable in the domain I, and $u = \{u_1, \dots, u_n\}$ is a generalized vector-function over* $C_0^\infty(I)$, satisfying equations (2), then in any subdomain $I' \subset I$ the generalized function u will be an ordinary infinitely differentiable function.

We shall say that an infinitely differentiable function $\varphi(x)$ in the domain I belongs to the class $\mathfrak{G}_q$ if there exist constants M and C such that

$$\max_{x \in I} |D^{(\alpha)} \varphi(x)| \leqslant M C^{(\alpha,q)} \Gamma((\alpha, q)), \tag{5}$$

where Γ is Euler' s gamma-function.

Theorem 2. *If the right-hand side and the coefficients of system (2) belong to the class $\mathfrak{G}_q$ in the domain I, and the vector-function $u(x)$ satisfies equations (2), then in any subdomain $I' \Subset I$ the vector-function $u(x)$ will be an infinitely differentiable function belonging to $\mathfrak{G}_q$.*

The proofs of Theorems 1 and 2 are based on a priori estimates. Theorem 1 is obtained by a modified method of paper (4). The derivation of Theorem 2 from the a priori estimates is carried out in the same way as the proof of analyticity of solutions of elliptic systems in (8), only instead of the embedding theorems of S. L. Sobolev one uses the embedding theorems of L. N. Slobodetskii (11).

4. Let us now dwell on a priori estimates for quasielliptic systems. We introduce the norms needed by us. We shall consider functions in the parallelepiped

$$I_R = \{x = (x_1, \dots, x_\nu),\ |x_i| \leqslant R^{q_i},\ i = 1, \dots, \nu\}.$$

By $I_r^{(i)}(y)$ we shall denote simply the segment $\{|x_i| \leqslant r^{q_i},\ x_k = y_k,\ k = 1, \dots, \dots, i-1, i+1, \dots, \nu\}$. Let $\varphi(x)$ be a smooth function, $0 < r < 1$. Following L. N. Slobodetskii, put

$$\|D_i^\gamma \varphi, I_r\|^2 = \iint_{I_r \times I_r^{(i)}} \frac{|\varphi(x) - \varphi(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_\nu)|^2}{|x_i - y_i|^{1+2\gamma}} \, dx \, dy_i.$$

If $\alpha = \beta + \gamma$, where β is an integral vector, and in the vector γ only the i-th coordinate γ_i is different from zero and $0 < \gamma_i < 1$, then put

$$\|D^\alpha \varphi, I_r\| = \|D_i^{\gamma_i}(D^\beta \varphi), I_r\|.$$

Denote by $A_{l,q}$ the set of all positive vectors $\alpha = (\alpha_1, \dots, \alpha_\nu)$ such that $(\alpha, q) = l$, all the numbers $\alpha_1, \dots, \alpha_\nu$, except possibly one, being integers. Put

$$\|\varphi, I_r\|_{l,q} = \left[\sum_{\alpha \in A_{l,q}} \|D^\alpha \varphi, I_r\|^2 \right]^{1/2}. \tag{6}$$

* By $C_0^\infty(I)$ is denoted the space of infinitely differentiable finite vector-functions in the domain I.

By means of the norm (6) one can introduce norms for the left- and right-hand sides of system (2), analogous to the norms (8). Choose numbers $s_1, \dots, s_n,\ t_1, \dots, t_n$ so that $0 \le s_i \le s$, and let $t = \max |t_i|$. Put

$$M_l(f) = \Gamma^{-1}(l+1) \sup_{R/2 \le r \le R} (R-r)^{s+t+l} \left[\sum_{i=1}^n \|f_i, I_r\|^2_{s-s_i+l,q} \right]^{1/2}, \tag{7}$$

$$N_l(u) = \Gamma^{-1}(l+1) \sup_{R/2 \le r \le R} (R-r)^{s+t+l} \left[\sum_{i=1}^n \|u_i, I_r\|^2_{s+t_i+l,q} \right]^{1/2}.$$

Theorem 3. Let, in the domain I_R, where $R < R_0$ (R_0 depends on the moduli of continuity of the coefficients $a_{ij}^{(\alpha)}(x)$), a smooth vector-function $u(x)$ satisfying (2) be given. Let l be a positive number, $l = N/\mu$ (N an integer). The inequality

$$N_l(u) \le K_0 M_l(f) + K_l \|u, I_r\|, \tag{8}$$

holds, where the constant K_0 does not depend on l; $\|u, I_r\|$ is the norm of the vector-function $u(x)$ in $L_2(I_R)$.

If the coefficients of system (1) belong to the class $\mathfrak{G}_q$, then the inequality

$$N_l(u) \le K \left[M_l(u) + \sum_{k=\mu^{-1}}^{p+s+t} C^k N_{p-k}(u) \right], \tag{9}$$

holds, where $\mu = m_1 \cdots m_\nu$, and the summation is over all fractions with denominator μ.

The proof of Theorem 3 is carried out in the usual way. Using the smallness of the domain, the operator $\mathcal{P}(x, D)$ is represented as the sum of an operator with constant coefficients and a small addition to it. First, by means of the Fourier transform, estimates are derived for the system of equations with constant coefficients. Then, by the method (8), the transition is made from constant coefficients to variable ones.

Received
2 VI 1960

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Note: Figure translations are in progress. See original paper for figures.

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