A NOTE ON THE APPLICATION OF SOLVABILITY CONDITIONS FOR THE CHAPLYGIN PROBLEM TO QUESTIONS IN THE QUALITATIVE THEORY OF EQUATIONS

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Abstract

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A NOTE ON THE APPLICATION OF SOLVABILITY CONDITIONS FOR THE CHAPLYGIN PROBLEM TO QUESTIONS IN THE QUALITATIVE THEORY OF EQUATIONS

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1. As has already been noted (see, for example, ([1–4])), the solvability conditions for the Chaplygin problem find broad application in the investigation of questions in the qualitative theory of equations. Of greatest interest, apparently, are solvability conditions of the type of Chaplygin' s theorem, i.e., conditions under which, for a solution y of the equation $x = F(x)$, where the operator F is defined in a certain partially ordered space, the inequality $z \preccurlyeq F(z)$ $(z \succcurlyeq F(z))$ entails the inequality $z \preccurlyeq y$ $(z \succcurlyeq y)$. In such a case we shall say that the operator F possesses property (M) (the monotonicity property).

The basic idea of applying solvability conditions of the type of Chaplygin' s theorem in qualitative theory may be formulated in the form of the following assertion.

Lemma. Let y_i $(i = 1, 2)$ be solutions of the equations $x_i = P_i(x_i)$, where the operators P_i map certain sets N_i into themselves, let the operator T map the product $N_1 \times N_2$ into a certain partially ordered set R, and let F be an operator in R possessing property (M) and satisfying the inequality

$$T[(y_1, P_2(x_2))] \preccurlyeq F(T[(y_1, x_2)]), \qquad x_2 \in N_2.$$

Then $T(y_1, y_2) \preccurlyeq y$, where y is a solution of the equation $x = F(x)$.

Below this lemma is applied in the investigation of Volterra' s integral equation.

2. Let X be a Banach space, also normed by elements of a certain KB-lineal Y ([5]). Everywhere elements of the space X are denoted by x, and elements of the space Y by y. In contrast to the usual norm, we shall denote the abstract norm of an element $x \in X$ by the symbol $|x|$, $x \in Y$. At the same time we assume that $\|x\| \leq \||x|\|$. By $G(b, u_0, r)$ we shall denote the topological product $[a, b) \times [a, b) \times S(u_0, r)$; here $S(u_0, r)$ is the ball of radius r with center at the point u_0, $r \leq \infty$, $b \leq \infty$.

We shall say that the operator $P[t,s,u]$, defined on $G(b,u_0,r)$, satisfies condition (A) if $P[t,s,u]$ can be represented in the form $P[t,s,u]=P_1[t,s,u]+P_2[t,s,u]$, where the operators $P_1[t,s,u]$ and $P_2[t,s,u]$ satisfy the conditions:

1) $\|P_1[t,s,u_2]-P_1[t,s,u_1]\| \le Q(t,s)\|u_2-u_1\|$, and

$$\lim_{t-t_1\to 0}\int_{t_1}^{t} Q(t,s)\,ds = \lim_{t-t_1\to 0}\int_{t_1}^{t} \|P_1[t,s,u_0]\|\,ds = 0.$$

2) $P_2[t,s,u]$ is an entirely continuous operator on $G(b,u_0,r)$.

3) $P_i[t,s,u]$ $(i=1,2)$ are continuous in t, uniformly with respect to s and u in each bounded subset $[a,b)\times S(u_0,r)$.

3. Consider in Y the equation

$$y(t)=\int_a^t K[t,\ s,\ y(s)]\,ds+\psi(t), \tag{1}$$

where the operator $K[t,\ s,\ y]$ is defined on $G(b,\ y_0,\ r)$, and $y(t)$ and $\psi(t)$ are functions with values in Y. From the theorem formulated in ([4]) there follows the following

Theorem 1. *Let the operator $K[t,\ s,\ y]$ satisfy condition (A) with operators $K_i[t,\ s,\ y]$ monotone in y $(i=1,2)$, let the function $\psi(t)$ be continuous, and let all solutions of equation (1) be defined on $[a,b)$, with the property that for each such solution $y(t)$ there exists only a finite number of continuous solutions of equation (1) that coincide with $y(t)$ on some interval $[a,\alpha)$.*

If a continuous function $z(t)$ satisfies the integral inequality

$$z(t)\leqslant \int_a^t K[t,\ s,\ z(s)]\,ds+\psi(t), \qquad t\in[a,b),$$

then $z(t)\leqslant y(t)$, where $y(t)$ is some solution of equation (1).

From this theorem, as a particular case, there follow, for example, the results of papers ([6,7]).

Let us now consider the integral equation in X

$$x(t)=\int_a^t P[t,\ s,\ x(s)]\,dx+f(t). \tag{2}$$

Suppose that the operator $P[t,\ s,\ x]$ satisfies condition (A) in $G(b,\ x_0,\ r)$. Suppose, moreover, that there exists an operator $K[t,\ s,\ y]$ defined in $G(b,\ |x_0-u|,\ r_1)$, where u is some element of X, such that

$$|P[t,\ s,\ x]| \leqslant K[t,\ s,\ |x-u|]$$

and that, for the corresponding equation (1), the hypotheses of Theorem 1 are fulfilled. The above-formulated solvability condition for Chaplygin' s problem then makes it possible to prove a number of comparison theorems for equation (1). For example, the following assertions hold.

Theorem 2. *Let $r_1 \leqslant r$ and $\psi(t) \geqslant |f(t)-u|$.*

Then:

1) Equation (2) has continuous solutions, and all these solutions are defined on $[a,b)$.

2) If $r=\infty$ and all solutions of equation (1) are bounded in norm, then all solutions of equation (2) are also bounded.

Below, for simplicity of formulation, we shall assume that

$$P[t,\ s,\ 0] \equiv f(t) \equiv x_0 = u = 0,$$

i.e., that $y(t) \equiv 0$ is a solution of equation (2). In this case we shall say that the solution $y(t) \equiv 0$ of equation (2) is **correct with respect to small perturbations of the operator and of the free term**, if for every $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that, whatever the operator $\widetilde{P}[t,\ s,\ x]$, satisfying in $G(b,\ 0,\ r_2)$ $(r_2 \leqslant r)$ condition (A) and the inequality

$$\left\| \int_a^t \widetilde{P}[t,\ s,\ x]\, ds \right\| < \delta_1,$$

and whatever the continuous function $\tilde{f}(t)$ on $[a,b)$, satisfying the inequality $\|\tilde{f}(t)\| < \delta_2$, the equation

$$x(t) = \int_a^t \{P[t,\ s,\ x(s)] + \widetilde{P}[t,\ s,\ x(s)]\}\, ds + \tilde{f}(t)$$

has continuous solutions on $[a,b)$, and all these solutions $x(t)$ satisfy the inequality $\|x(t)\| < \varepsilon,\ t \in [a,b)$.

Theorem 3. *Let $K[t,\ s,\ 0] \equiv \psi(t) \equiv 0$.*

Then:

1) If equation (1) has a unique solution in $S(0,r_1)$, then equation (2) also has a unique solution in $S(0,r_1)$.

2) If the zero solution of equation (1) is well-posed with respect to small perturbations of the free term, then the zero solution of equation (2) is well-posed with respect to small perturbations of the operator and of the free term.

We note that if $b = \infty$, then well-posedness of the solution means its stability. In this case, from the asymptotic stability of the zero solution of equation (1) there follows the asymptotic stability of the zero solution of equation (2).

Theorems 2 and 3 generalize some results of the works ([1,8,9]).

Remark. Theorems 2 and 3 were obtained on the basis of the lemma with $T(x_1, x_2) = |x_1 - x_2|$. It is easy to see that, under a corresponding change of the conditions, these results are also valid for another choice of $T(x_1, x_2)$.

We also note that Theorems 2 and 3 are valid for certain classes of operator equations of the form

$$x(t) = F[t, x(\cdot)].$$

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Note: Figure translations are in progress. See original paper for figures.