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Abstract

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER IN A BANACH SPACE WHEN THE SMALL PARAMETER AT THE HIGHEST DERIVATIVE TENDS TO ZERO

(Presented by Academician S. L. Sobolev on 29 VI 1960)

Let Y be a Banach space. The space of linear bounded operators mapping the space Y into Y will be denoted by $\{Y \to Y\}_{[1]}$. Consider the ordinary linear differential equation of second order in the Banach space Y:

$$\varepsilon y'' + p_0(x)y' + q(x)y = a(x). \tag{1}$$

Here $\varepsilon > 0$ is a small real parameter tending to zero; $y = y(x, \varepsilon) \in Y$ is the unknown function of the real variable x; $p_0(x) \geq \chi > 0$ is a real function, $q(x) \in \{Y \to Y\}$; $a(x) \in Y$ (the functions p_0, q, a are defined for $x_1 \leq x \leq x_2$). Suppose that the functions $p_0(x)$, $q(x)$, $a(x)$ are continuous, infinitely differentiable, and

$$\|q(x)\| \leq K, \qquad \|a(x)\| \leq K, \qquad \|q^{(n)}(x)\| \leq M(n), \qquad \|a^{(n)}(x)\| \leq M(n)$$
$$(x_1 \leq x \leq x_2), \tag{2}$$

where $K, M(n)$ are constants ($M(n)$ may increase together with n). Under these assumptions it will be proved that any solution $y(x, \varepsilon)$ of equation (1) with Cauchy initial data tends, as $\varepsilon \to 0$, to the corresponding solution of the limiting equation

$$p_0(x)y' + q(x)y = a(x). \tag{3}$$

The method of investigating equation (1) is based on the theory of asymptotic series in a Banach space ([2]). As is known ([2]), for an arbitrary set of elements $C_0, C_1, C_2, ... \in Y$ there exists a function $f(\varepsilon) \in Y$ (and consequently infinitely many such functions) which satisfies the asymptotic relation $f(\varepsilon) \sim \sum_{i=0}^{\infty} C_i \varepsilon^i$.

Taking an arbitrary set of functions $C_0(x), C_1(x), C_2(x), ... \in Y$, one can construct such a function $f(x,\varepsilon) \in Y$ that the relation

$$f(x,\varepsilon) \sim \sum_{i=0}^{\infty} C_i(x)\varepsilon^i \tag{4}$$

will hold.

It is obvious that this asymptotic relation can be differentiated with respect to x, if the coefficients $C_i(x)$ are differentiable. We shall first consider the homogeneous equation

$$\varepsilon y'' + p_0(x)y' + q(x)y = \theta. \tag{5}$$

Simultaneously with equation (5) we shall consider the equation

$$\varepsilon \bar{y}'' + p_0(x)\bar{y}' + \bar{q}(x)\bar{y} = \bar{\theta}. \tag{6}$$

in the space $\{Y \to Y\}$. Obviously, knowing some solution $\bar{y} = \bar{y}(x,\varepsilon)$ of equation (6), we obtain a solution $y = \bar{y}(x,\varepsilon)C$ of equation (5), where $C \in Y$.

Theorem 1. Equation (6) has two independent solutions $\bar{y}_1(x,\varepsilon)$, $\bar{y}_2(x,\varepsilon)$, satisfying the relations

$$\bar{y}_1(x,\varepsilon) \sim \exp\left[-\frac{1}{\varepsilon}\int_{x_0}^{x} p_0(t)\,dt\right] \sum_{i=0}^{\infty} \bar{u}_{1i}(x)\varepsilon^i, \tag{7}$$

$$(x_1 \leqslant x_0 \leqslant x \leqslant x_2)$$

$$\bar{y}_2(x,\varepsilon) \sim \sum_{i=0}^{\infty} \bar{u}_{2i}(x)\varepsilon^i, \tag{8}$$

where the functions $\bar{u}_{1i}(x)$, $\bar{u}_{2i}(x) \in \{Y \to Y\}$ are continuous, infinitely differentiable, and moreover

$$\bar{y}_1(x_0,\varepsilon) = I, \qquad \bar{y}_2(x_0,\varepsilon) = I \quad (I \text{ is the identity operator}). \tag{9}$$

The solutions $\bar{y}_1, \bar{y}_2$ are independent in the sense that the function

$$y(x,\varepsilon) = \bar{y}_1(x,\varepsilon)C_1 + \bar{y}_2(x,\varepsilon)C_2 \quad (C_1, C_2 \in Y;\ x_0 \leqslant x \leqslant x_2) \tag{10}$$

will be the general solution of equation (5).

Proof. Substituting (7), (8) into (6), we obtain:

$$-p_0\bar{u}_{10}'+q\bar{u}_{10}=\bar{\theta}, \qquad -p_0\bar{u}_{11}'+q\bar{u}_{11}+\bar{u}_{10}''=\bar{\theta},\dots,-p_0\bar{u}_{1i}'+q\bar{u}_{1i}+\bar{u}_{1\,i-1}''=\bar{\theta},\dots \quad (11)$$

$$p_0\bar{u}_{20}'+q\bar{u}_{20}=\bar{\theta}, \qquad p_0\bar{u}_{21}'+q\bar{u}_{21}+\bar{u}_{20}''=\bar{\theta},\dots,p_0\bar{u}_{2i}'+q\bar{u}_{2i}+\bar{u}_{2\,i-1}''=\bar{\theta},\dots \quad (12)$$

We shall assume that the solutions $\bar{u}_{1i}(x)$, $\bar{u}_{2i}(x)$ $(i=0,1,2,\dots)$ of equations (11), (12) satisfy the initial conditions: for $x=x_0$

$$\bar{u}_{k0}=I, \qquad \bar{u}_{ki}=\bar{\theta} \quad (k=1,2;\ i=1,2,\dots). \quad (13)$$

On the basis of (4) we obtain: there exist such functions $\bar{y}_1^*,\bar{y}_2^*$ that

$$\bar{y}_1^*(x,\varepsilon)\sim\exp\left[-\frac{1}{\varepsilon}\int_{x_0}^{x}p_0(t)\,dt\right]\sum_{i=0}^{\infty}\bar{u}_{1i}(x)\varepsilon^i, \qquad \bar{y}_2^*(x,\varepsilon)\sim\sum_{i=0}^{\infty}\bar{u}_{2i}(x)\varepsilon^i, \quad (14)$$

and moreover $\bar{y}_1^*(x_0,\varepsilon)=\bar{y}_2^*(x_0,\varepsilon)=I$. The functions $\bar{y}_1^*(x,\varepsilon)$, $\bar{y}_2^*(x,\varepsilon)$ in general are not solutions of equation (6), but with their help we shall show the existence of the desired solutions $\bar{y}_1(x,\varepsilon)$, $\bar{y}_2(x,\varepsilon)$. To prove this, introduce

$$\bar{z}_k=\bar{y}_k-\bar{y}_k^*(x,\varepsilon) \quad (k=1,2), \quad (15)$$

where $\bar{y}_k$ is a solution of equation (6) satisfying the relations

$$\bar{y}_k(x_0,\varepsilon)=I, \qquad \bar{y}_k'(x_0,\varepsilon)=\bar{y}_k^{*\prime}(x_0,\varepsilon). \quad (16)$$

Substituting $\bar{y}_k=\bar{z}_k+\bar{y}_k^*$ into (6), we obtain for $\bar{z}_k$ the differential equation

$$\varepsilon\bar{z}_k''+p_0(x)\bar{z}_k'+q(x)\bar{z}_k=\bar{\alpha}_k(x,\varepsilon), \quad (17)$$

where

$$\bar{\alpha}_k(x,\varepsilon)=-\varepsilon\bar{y}_k^{*\prime\prime}-p_0(x)\bar{y}_k^{*\prime}-q(x)\bar{y}_k^*, \quad (18)$$

moreover, $\bar{z}_k$, on the basis of (15), (16), satisfies the initial conditions: for $x=x_0$

$$\bar{z}_k=\bar{\theta}, \qquad \bar{z}_k'=\bar{\theta}. \quad (19)$$

Since $\bar{y}_k^*(x,\varepsilon)$ has the asymptotic expansion (14), which satisfies (6) formally, it follows from (18) that $\bar{\alpha}_k(x,\varepsilon) \sim \bar{\theta}$, or

$$\|\bar{\alpha}_k(x,\varepsilon)\| \leqslant \varepsilon^{N+m} \tag{20}$$

for arbitrary positive numbers N, m and all sufficiently small $\varepsilon > 0$ ($\varepsilon \to 0$). We write the solution $\bar{z}_k$ of equation (17) under condition (19) in the form of a series

$$\bar{z}_k(x,\varepsilon) = \sum_{m=1}^{\infty} \bar{u}_{km}(x,\varepsilon), \tag{21}$$

where

$$\bar{u}_{k1}(x,\varepsilon) = \frac{1}{\varepsilon}\int_{x_0}^{x}\int_{x_0}^{\tau_1} \bar{\alpha}(\tau_2,\varepsilon)\, d\tau_2 d\tau_1, \ldots$$

$$\ldots, \bar{u}_{k,m+1}(x,\varepsilon) = -\frac{1}{\varepsilon}\int_{x_0}^{x}\int_{x_0}^{\tau_1} \left[p_0(\tau_2)\bar{u}_{km}(\tau_2,\varepsilon) + q(\tau_2)\bar{u}_{km}(\tau_2,\varepsilon)\right] d\tau_2 d\tau_1, \ldots \tag{22}$$

Hence, and from (2), (20), we obtain

$$\|\bar{u}_{k,m+1}\| \leqslant (2K)^m \frac{(x-x_0)^{m+1}}{(m+1)!}\varepsilon^N.$$

Consequently, the solution (21) satisfies the inequality $\|\bar{z}_k(x,\varepsilon)\| \leqslant L\varepsilon^N$ ($L =$ const) for all sufficiently small $\varepsilon > 0$, i.e.

$$\bar{z}_k(x,\varepsilon) \sim \bar{\theta}, \qquad x_1 \leqslant x \leqslant x_2 \quad (k = 1, 2). \tag{23}$$

Hence (7) and (8) follow from (14).

Let y_0, y_0' be any prescribed elements of Y. For the solution $y = y(x,\varepsilon)$ of equation (5), from (10) we have: for $x = x_0$

$$y = C_1 + C_2 = y_0, \qquad y' = \bar{y}_1'(x_0,\varepsilon)C_1 + \bar{y}_2'(x_0,\varepsilon)C_2 = y_0'. \tag{24}$$

Hence

$$[\bar{y}_2'(x_0,\varepsilon) - \bar{y}_1'(x_0,\varepsilon)]C_1 = \bar{y}_2'(x_0,\varepsilon)y_0 - y_0',$$

$$[\bar{y}_2'(x_0, \varepsilon) - \bar{y}_1'(x_0, \varepsilon)]C_2 = y_0' - \bar{y}_1'(x_0, \varepsilon)y_0. \tag{25}$$

From (7), (8) we have

$$\bar{y}_1'(x_0, \varepsilon) = -\frac{1}{\varepsilon}p_0(x) + \bar{u}_{10}'(x_0) + \bar{\eta}_1(\varepsilon),$$

$$\bar{y}_2'(x_0, \varepsilon) = \bar{u}_2'(x_0) + \bar{\eta}_2(\varepsilon),$$

where $\bar{\eta}_1(\varepsilon), \bar{\eta}_2(\varepsilon)$ tend to $\bar{\theta}$ as $\varepsilon \to 0$. Consequently, for all sufficiently small ε the operator $\bar{y}_2'(x_0, \varepsilon) - \bar{y}_1'(x_0, \varepsilon)$ has an inverse

$$[\bar{y}_2'(x_0, \varepsilon) - \bar{y}_1'(x_0, \varepsilon)]^{-1} = \varepsilon\bar{\gamma}(\varepsilon),$$

where $\bar{\gamma}(\varepsilon)$ is a bounded function, with $\bar{\gamma}(\varepsilon) \to 1/p_0(x_0)$ as $\varepsilon \to 0$. Hence, from (25), we obtain

$$C_1 = \varepsilon\bar{\gamma}(\varepsilon)(\bar{y}_2'(x_0, \varepsilon)y_0 - y_0'),$$

$$C_2 = \varepsilon\bar{\gamma}(\varepsilon)(y_0' - \bar{y}_1'(x_0, \varepsilon)y_0),$$

and $C_1 \to \theta,\ C_2 \to y_0$ as $\varepsilon \to 0$.

Theorem 2. *The solution $y(x, \varepsilon)$ of equation (5) $(x_0 \leqslant x \leqslant x_2)$, satisfying the initial conditions (24), tends as $\varepsilon \to 0$ to the solution $y_0(x)$ of the degenerate equation $p_0(x)y' + q(x)y = 0$, satisfying the condition $y_0(x_0) = y_0$.*

Indeed, in (10), $\bar{y}_1(x, \varepsilon)C_1 \sim \theta$, $\bar{y}_2(x, \varepsilon)C_2 \to \bar{u}_{20}(x)y_0 = y_0(x)$, and

$$y_0(x_0) = \bar{u}_{20}(x_0)y_0 = y_0.$$

Theorem 3. The solution $y(x, \varepsilon)$ of equation (1), satisfying the initial conditions (24), tends, as $\varepsilon \to 0$, to the solution $y_0(x)$ of the degenerate equation (3), satisfying the condition $y_0(x_0) = y_0$. The derivatives of the solution $y(x, \varepsilon)$ behave analogously for $x_0 < x \leqslant x_2$ $(x \neq x_0)$.

Proof. Obviously, if $\tilde{y}(x, \varepsilon)$ is a particular solution of equation (1), satisfying the conditions $\tilde{y}(x_0, \varepsilon) = \theta$, $\tilde{y}'(x_0, \varepsilon) = \tilde{y}_0$ ($\tilde{y}_0$ is some element of Y), then

$$y(x, \varepsilon) = \bar{y}_1(x, \varepsilon)C_1 + \bar{y}_2(x, \varepsilon)C_2 + \tilde{y}(x, \varepsilon) \tag{26}$$

will be the general solution of equation (1). Substituting $\sum_{i=0}^{\infty} \tilde{u}_i(x)\varepsilon^i$ into (1), we obtain the system

$$p_0\tilde{u}_0' + q\tilde{u}_0 = a(x), \qquad p_0\tilde{u}_1' + q\tilde{u}_1 + \tilde{u}_0'' = \theta, \dots, \quad p_0\tilde{u}_i' + q\tilde{u}_i + \tilde{u}_{i-1}'' = \theta, \dots \quad (27)$$

with $\tilde{u}_i(x_0) = \theta$ $(i = 0, 1, 2, \dots)$. Analogously to the preceding case (Theorem 1), it is proved that there exists a solution $\tilde{y}(x, \varepsilon)$ of equation (1), satisfying the condition

$$\tilde{y}(x, \varepsilon) \sim \sum_{i=0}^{\infty} \tilde{u}_i(x)\varepsilon^i, \quad (28)$$

and $\tilde{y}(x_0, \varepsilon) = \theta$, $\tilde{y}'(x_0, \varepsilon) = \tilde{y}_0$. C_1, C_2 in (26) are determined from the equations

$$C_1 + C_2 = y_0, \qquad \bar{y}_1'(x_0, \varepsilon)C_1 + \bar{y}_2'(x_0, \varepsilon)C_2 = y_0' - \tilde{y}_0'. \quad (29)$$

Further, analogously to the preceding case (Theorems 1 and 2), it is proved that in (26) $\bar{y}_1(x, \varepsilon)C_1 \sim \theta$, $\bar{y}_2(x, \varepsilon)C_2 + \tilde{y}(x, \varepsilon) \to y_0(x)$, i.e. $y(x, \varepsilon) \to y_0(x)$, $x_0 \leqslant x \leqslant x_2$ (for $x = x_0$, $\|y'\| \to \infty$).

Example 1. Let Y be the space $C_{[a,b]}$ of continuous functions $y = v(t)$. Equation (1) takes the form

$$\varepsilon \frac{\partial^2 y(x,t)}{\partial x^2} + p_0(x) \frac{\partial y(x,t)}{\partial x} + \int_a^b Q(x, t; \tau)\, d\tau = A(x, t),$$

where the functions $p_0(x) \geqslant \varkappa > 0$, $Q(x, t; \tau)$, $A(x, t)$ are continuous for $x_1 \leqslant x \leqslant x_2$, $a \leqslant t, \tau \leqslant b$, infinitely differentiable with respect to x for $x_1 \leqslant x \leqslant x_2$; $\varepsilon \to 0$ $(\varepsilon > 0)$.

Example 2. Let Y be an l-dimensional space. Equation (1) is transformed into the system

$$\varepsilon y_j'' + p_0(x) y_j' + \sum_{i=1}^{l} q_{ji}(x) y_i = a_j(x) \qquad (j = 1, 2, \dots, l),$$

where the functions $p_0(x) \geqslant \varkappa > 0$, $q_{ji}(x)$, $a_j(x)$ $(j, i = 1, 2, \dots, l)$ are continuous and infinitely differentiable for $x_1 \leqslant x \leqslant x_2$; $\varepsilon \to 0$ $(\varepsilon > 0)$.

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References

1. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, Moscow-Leningrad, 1951.

2. L. A. Groza, DAN, **121**, No. 6 (1958).

Note: Figure translations are in progress. See original paper for figures.