MATHEMATICS

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Abstract

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MATHEMATICS

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ON ECONOMICAL IMPLICIT SCHEMES

(THE METHOD OF FRACTIONAL STEPS)

(Presented by Academician S. L. Sobolev on 28 V 1960)

1. Consider a system of linear equations written in matrix form:

$$\frac{\partial u}{\partial t} = P(D)u. \tag{1}$$

Here $P(D)$ is a matrix acting on the vector u, whose elements are polynomials in the differentiation symbols $D_i = \partial/\partial x_i$, $i = 1, \dots, m$. In index notation equation (1) is written as

$$\frac{\partial u_i}{\partial t} = \sum_{j,\alpha} a_{ij\alpha_1 \dots \alpha_m} D_1^{\alpha_1} \cdots D_m^{\alpha_m} u_j. \tag{2}$$

As is known, a **homogeneous difference scheme** is a scheme invariant with respect to shift operators. Such a scheme can be written in the form

$$\Lambda(T)u = 0, \tag{3}$$

where $\Lambda(T)$ is a polynomial in the shift operators

$$\Lambda(T) = \sum_{\alpha} a_{\alpha_0\alpha_1 \dots \alpha_m} T_0^{\alpha_0} T_1^{\alpha_1} \cdots T_m^{\alpha_m}, \tag{4}$$

$$T_i^{\alpha_i} f(x_0, x_1, \dots, x_m) = f(x_0, \dots, x_i + \alpha_i h_i, \dots, x_m), \qquad x_0 = t. \tag{5}$$

The most general homogeneous two-layer scheme solving equation (1) has the form

$$\frac{u^{n+1} - u^n}{\tau} = \Pi_1(T)u^{n+1} + \Pi_2(T)u^n, \tag{6}$$

where

$$\Pi(T) \sim P(D), \qquad \Pi(T) = \Pi_1(T) + \Pi_2(T) \tag{7}$$

($\sim$ is the sign of approximation),

$$u^n(x_1, \dots, x_m) = u(n\tau, x_1, \dots, x_m). \tag{8}$$

When $\Pi_1(T) \equiv 0$, scheme (6) is explicit and the solution algorithm is simple and obvious. However, as a rule, explicit schemes are conditionally stable, i.e. stable for sufficiently small τ. Therefore one resorts to absolutely stable schemes, which, as a rule, are implicit, so that $\Pi_1(T) \neq 0$. With an increase in the number m of unknowns and in the order of the poly-

the dimension of the operator $\Pi_1(T)$, which leads to a complication of the scheme and to an increase in the number of operations. This is the main drawback of homogeneous schemes: in solving the problem of constructing a stable and approximating scheme at once in a single step, we inevitably and to a considerable degree complicate the solution algorithm.

2. Consequently, in order to solve this problem without greatly complicating the algorithm, we must solve it in stages, introducing auxiliary fractional steps. At each fractional step the approximation and stability of the difference operator need not hold, being attained only in the transition from an integral step to an integral one.

Recently a number of schemes based on this principle have appeared ($^{1-6}$). We shall formulate a general theorem that makes it possible to establish convergence in general form for all schemes of this type.

Theorem. *Let the scheme with fractional steps*

$$\frac{u^{n+s/p} - u^{n+(s-1)/p}}{\tau} = \Pi_{1s}(T)u^{n+s/p} + \Pi_{2s}(T)u^{n+(s-1)/p}, \quad s = 1, \dots, p, \tag{9}$$

satisfy the conditions

$$\Pi_{1s}(T) \sim P_{1s}(D), \quad \Pi_{2s}(T) \sim P_{2s}(D), \quad \sum_{s=1}^{p} (P_{1s} + P_{2s}) = P. \tag{10}$$

Each of the two-layer schemes (9) has a multiplication matrix g_s (^{7}), such that

$$\|g_1 g_2 \dots g_p\| = 1 + O(\tau). \tag{11}$$

Then the solution of (9) converges to the solution of (1) in the mean.

Proof. We first give the proof for the case where u is a scalar, and not a vector function. Write equations (9) in the form:

$$A_s u^{n+s/p} - B_s u^{n+(s-1)/p} = 0, \tag{12}$$

$$A_s = E - \tau\Pi_{1s}, \quad B_s = E + \tau\Pi_{2s}. \tag{13}$$

Using the commutativity of the operators A_s, B_s, we obtain an equivalent homogeneous scheme containing no fractional steps:

$$A_1 \cdot A_2 \cdot \ldots \cdot A_p u^{n+1} - B_1 \cdot B_2 \cdot \ldots \cdot B_p u^n = 0. \tag{14}$$

Expanding (14) in powers of τ, after simple transformations

$$\frac{u^{n+1} - u^n}{\tau} = \sum_{s=1}^{p} \Pi_{1s} u^{n+1} + \sum_{s=1}^{p} \Pi_{2s} u^n + O(\tau) = \sum_{s=1}^{p} (\Pi_{1s} + \Pi_{2s}) u^n + O(\tau). \tag{15}$$

By virtue of (10), scheme (15), and hence the scheme (9) equivalent to it, approximate equation (1). By virtue of (11), the schemes (15), (9) are stable, as was required to prove.

Let u be a vector function. We continue the equalities (9), replacing p by $p+1$:

$$\frac{u^{n+s/p+1} - u^{n+(s-1)/p+1}}{\tau} = \Pi_{1s} u^{n+s/p+1} + \Pi_{2s} u^{n+(s-1)/p+1}. \tag{16}$$

Adding first the p equalities (9), then the last $p-1$ equalities (9) with the first (16), the last $p-2$ equalities (9) with the first two (16)

and continuing analogously, we obtain the equalities:

$$\frac{u^{n+s/p+1} - u^{n+s/p}}{\tau} =$$

$$= \sum_{r=1}^{s} \Pi_{1r} u^{n+r/p+1} + \sum_{r=s+1}^{p} \Pi_{1r} u^{n+r/p} + \sum_{r=1}^{s} \Pi_{2r} u^{n+(r-1)/p+1} + \sum_{r=s+1}^{p} \Pi_{2r} u^{n+(r-1)/p}. \tag{12'}$$

Considering the quantities $u^{n+1/p}, u^{n+2/p}, \ldots, u^{n+p/p}$ as unknown functions $v_1^n, v_2^n, \ldots, v_p^n$, we see that the difference equations (12) approximate the system of differential equations

$$\frac{\partial v_i}{\partial t} = \sum_{r=1}^{p} \Pi_{1r} v_r + \Pi_{21} v_p + \sum_{r=2}^{p} \Pi_{2r} v_{r-1}. \tag{17}$$

By virtue of (11), the difference system (12) and the equivalent system (12′) are stable. Consequently, the differential system (17) is stable, i.e., the Cauchy problem for (17) is well posed. For $v_1 = v_2 = \cdots = v_p = u$, the system (17) passes into system (1) and, consequently, approximates the latter. Hence it follows that the solution of system (12) converges to the solution of (1), as was required to prove.

3. The advantage of the method described (the method of fractional steps) consists in the fact that the dimensions of the operators $\Pi_{1s}(T), \Pi_{2s}(T)$ are smaller than the dimensions of the operators $\Pi_1(T), \Pi_2(T)$. Thus the implicit schemes used at the fractional steps are considerably simplified and in many cases can be reduced to implicit three-point schemes solved by sweeping. This simplification of implicit schemes more than compensates for the increase in the amount of computation due to the introduction of fractional steps. It is clear that the method of fractional steps can be applied to the construction of iterative schemes for solving certain systems of linear equations.

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