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## Abstract

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### MATHEMATICS

V. P. IL' IN

# ON THE COMPLETE CONTINUITY OF THE EMBEDDING OPERATOR FOR THE CASE OF AN UNBOUNDED DOMAIN

*(Presented by Academician S. L. Sobolev on 17 VI 1960)*

In the present note we shall give some sufficient conditions for the complete continuity of the embedding operator for the case when the functions of the corresponding functional classes are defined on the whole $n$-dimensional space $E_n$ or on a subspace $E_m$ of dimension $m < n$. Questions of a similar kind for the special case (embedding in $L_2(E_n)$) were studied by A. M. Molchanov ([1]), M. Sh. Birman ([2]), and M. Sh. Birman and B. S. Pavlov*. The results of these authors have the character of necessary and sufficient conditions.

Let $E_n$ be $n$-dimensional Euclidean space, and let $\omega(X) = \omega(x_1, \dots, x_n)$ be a positive measurable function given in $E_n$. Let $p > 1$.

Denote by $L_p(\omega; E_n)$ the set of functions $f(X)$, defined in $E_n$, for which

$$\|f\|_{L_p(\omega;E_n)} = \left[\int_{(E_n)} \cdots \int \omega |f|^p \, dX\right]^{1/p} < \infty. \tag{1}$$

Let $l$ be an arbitrary positive number, $\bar{l} = [l]$. Suppose that $f(X)$ has continuous derivatives of order $\bar{l}$. Put

$$\|f\|_{L_p^{(l)}(E_n)} = \sum_{i_1,\dots,i_{\bar{l}}=1}^{n} \left[\int_{(E_n)} \cdots \int \left(\int_{(E_n)} \cdots \int \frac{\left|\dfrac{\partial^{\bar{l}} f(X)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}} - \dfrac{\partial^{\bar{l}} f(Y)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}}\right|^p}{|X-Y|^{n+(l-\bar{l})p}} \, dY\right) dX\right]^{1/p} \tag{2}$$

when $l$ is not an integer, and

$$\|f\|_{L_p^{(l)}(E_n)} = \sum_{i_1,\dots,i_l=1}^{n} \left[ \int_{(E_n)} \cdots \int \left| \frac{\partial^l f(X)}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^p dX \right]^{1/p} \tag{2'}$$

when $l$ is an integer, if the right-hand sides of (2) or, respectively, (2′) are finite. Further, put

$$\|f\|_{W_p^{(l)}(\omega;E_n)} = \|f\|_{L_p(\omega;E_n)} + \|f\|_{L_p^{(l)}(E_n)}. \tag{3}$$

If $\omega \equiv 1$, we shall write $W_p^{(l)}(E_n)$, respectively $L_p(E_n)$. We define the space $W_p^{(l)}(\omega; E_n)$ as the closure, in the norm (3), of the set of $\bar{l}$-times continuously differentiable functions for which the right-hand side of (3) is finite.

* Unpublished.

Further, by $\overset{(n)}{R}(X)$ we shall denote the $n$-dimensional ball of radius $R$ with center at $X$; in particular, $\overset{(n)}{R}(0)$ denotes the ball with center at the origin; $E_n - \overset{(n)}{R}(0)$ is the complement of $\overset{(n)}{R}(0)$ in $E_n$; $|X| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the distance from $X$ to the origin.

**Theorem 1.** Let $F$ be a set of functions $f(X) \in W_p^{(l)}(\omega; E_n)$ such that

$$\sup_{f\in F} \|f\|_{W_p^{(l)}(\omega;E_n)} \le M, \tag{4}$$

where $\omega(X)$ satisfies the conditions:

$$\sup_{Y\in E_n} \int_{\overset{(n)}{H}(Y)} \cdots \int \omega^{-p'/p}(X)\,dX < \infty; \qquad \int_{\overset{(n)}{H}(Y)} \cdots \int \omega^{-p'/p}(X)\,dX \to 0 \quad \text{as } |Y| \to \infty, \tag{5}$$

where $H > 0$ is a fixed number.

Then, if

$$0 \le s < l, \qquad q \ge p > 1, \qquad m \text{ is an integer}, \qquad 1 \le m \le n, \qquad l - s + \frac{m}{q} - \frac{n}{p} > 0, \tag{6}$$

the functions $f \in F$, considered on the hyperplane $E_m$ of dimension $m$, form a set $F'$ compact in $W_q^{(s)}(E_m)$.

**Proof.** Starting from the inequality

$$\left|\frac{\partial^k f(X)}{\partial x_{i_1}\cdots\partial x_{i_k}}\right| \le \frac{C_1}{h^{n+k}}\int_{\overset{(n)}{h}(X)}\cdots\int |f(Y)|\,dY +$$

$$+\sum_{i_1,\dots,i_l=1}^{n} C_{i_1\dots i_l}\int_0^h \frac{dv}{v^{l+n+k-l}}\left[\int_{\overset{(n)}{v}(X)}\cdots\int\left(\int_{(E_n)}\cdots\int\left|\frac{\frac{\partial^l f(Y)}{\partial x_{i_1}\cdots\partial x_{i_l}}-\frac{\partial^l f(Z)}{\partial z_{i_1}\cdots\partial z_{i_l}}}{|Y-Z|^{n+(l-l)/p}}\right|^p dZ\right)^{1/p} dY\right]$$

(for integral $l$ the second term is written more simply), which is valid for continuously differentiable functions for every integer $k < l$, where $h$ is an arbitrary positive number, the $C_i$ do not depend on $f$ and $h$, and also from analogous inequalities for differences of derivatives of order $k$, it is not difficult to prove that:

1) For any domain $\Omega_n \subset E_n$ (in particular, for $\Omega_n = E_n$) the inequality

$$\|f\|_{L_p(\Omega_n)} \le C_2 h^{-n/p'}\|f\|_{L_p(\omega;E_n)}\sup_{X\in\Omega_n}\left(\int_{\overset{(n)}{h}(X)}\cdots\int \omega^{-p'/p}(Y)\,dY\right)^{1/p'} + C_3 h^l \|f\|_{L_p^{(l)}(E_n)} \tag{7}$$

holds;

2)

$$\|f\|_{W_q^{(s)}(E_m)} \le C_4\left(\|f\|_{L_p(E_n)}^{l-s+m/q-n/p}\,\|f\|_{W_p^{(l)}(\omega;E_n)}^{n/p+s-m/q}\right)^{1/l}. \tag{8}$$

It follows from (8) ($l-s+m/q-n/p>0$) that it is sufficient to prove the compactness of the set $F$ in $L_p(E_n)$; and for this it is necessary and sufficient that, for every $R>0$, $F$ be compact in the metric $L_p(\overset{(n)}{R}(0))$ and that, as $R\to\infty$,

$$\sup_{f\in F}\|f\|_{L_p(E_n-\overset{(n)}{R}(0))}\to 0. \tag{9}$$

The first assertion is proved in the usual way, and the second follows from (8) for $\Omega_n = E_n - \overset{(n)}{R}(0)$. Indeed, taking $h \le H$, on the basis of (4) we have:

$$\|f\|_{L_p(E_n-\overset{(n)}{R}(0))} \le C_5 M\left[h^{-n/p'}\sup_{X\in E_n-\overset{(n)}{R}(0)}\left(\int_{\overset{(n)}{H}(X)}\cdots\int\omega^{-p'/p}(Y)\,dY\right)^{1/p'} + h^l\right],$$

whence, by virtue of (5) and the arbitrariness of $h$, we obtain (9).

**Remark.** Put, for any positive number $s$,

$$\|f\|_{C^{(s)}(E_n)} = \sup_{X\in E_n}|f(X)| + \sup_{X\in E_n}|f'(X)| + \cdots + \sup_{X\in E_n}|f^{[s]}(X)| + \sup_{X,Y\in E_n}\frac{|f^{[s]}(X)-f^{[s]}(Y)|}{|X-Y|^{s-[s]}}.$$

Then, if condition (6) of Theorem 1 is fulfilled in the form

$$0 \le s < l, \qquad l - s - \frac{n}{p} > 0, \tag{6'}$$

then the set $F$ is compact in $C^{(s)}(E_n)$.

**Theorem 2.** Let $F$ be a set of functions $f(X) \in W_p^{(l)}(E_n)$ for which

$$\|f\|_{W_p^{(l)}(E_n)} \le M. \tag{5'}$$

Let $\bar{s}, m$ be integers, $0 \le \bar{s} < l$, $1 \le m \le n$, $q \ge p > 1$, $l - \bar{s} + m/q - n/p > 0$. Further, let $E_m$ be a hyperplane of dimension $m$; let $\omega(X)$ be a positive function defined on $E_m$, satisfying the conditions:

1) $\sup\limits_{Y\in E_m}\int_{\overset{(m)}{H}(Y)}\cdots\int \omega(X)\,dX < \infty, \quad \int_{\overset{(m)}{H}(Y)}\cdots\int \omega(X)\,dX \to 0 \quad \text{as } |Y|\to\infty;$
2) there exists a number $\delta > 0$ such that

$$\sup_{Y\in E_m}\left(\int_{\overset{(m)}{H}(Y)}\cdots\int \frac{\omega(X)\,dX}{|X-Y|^{[n/p+s+\delta-l]q}}\right) < \infty, \tag{10-11}$$

where $H > 0$ is a fixed number.

Then the derivatives of order $\bar{s}$ of the functions $f \in F$, considered on the hyperplane $E_m$, form a set $F'$ compact in $L_q(\omega; E_m)$.

For the proof the same methods as above are applied.

**Remark.** If $l - \bar{s} - n/p > 0$, then condition (11) is superfluous (one may put $\delta = l - \bar{s} - n/p > 0$, and then (11) coincides with the first of conditions (10)). In this case M. Sh. Birman and B. S. Pavlov obtained necessary and sufficient conditions.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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2. M. Sh. Birman, *DAN*, **125**, No. 3 (1959).

*Note: Figure translations are in progress. See original paper for figures.*