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Abstract

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THE BRANCHING EQUATION AND THE NEWTON DIAGRAM

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The note gives a new derivation of the Lyapunov-Schmidt branching equation (^{1,2}) for an equation in spaces of type B, making it possible to include nonlinear singular integral equations within the range of the branching theories under consideration. On the other hand, the Newton diagram (^{3}), used by a number of authors to study the branching equation ($^{4-6}$), is proposed for direct application to the functional equation, without forming the branching equation, which, it seems to us, is advantageous in practical computations. The analytic case is also considered. Here, after finding all possible values $\varepsilon > 0$ by means of the Newton diagram, the method of N. N. Nazarov (^{7,9}) is convenient for practical calculation. The note uses the methods of the author' s papers (^{9,10}).

Consider the equation

$$F(x, y) = 0, \tag{1}$$

where x, y, and $F(x, y)$ belong respectively to the Banach spaces E, E_1, and E_2 (real or complex).

Let $F(x_0, y_0) = 0$; we seek solutions $y = y(x)$ of equation (1), defined in some c-star about the point x_0 and tending to y_0 as $x = x_0$.

Definition (I. Ts. Gokhberg and M. G. Krein). We shall call a linear operator $B \in \{E_1 \to E_2\}$ an F-operator if: 1) the range of the operator B is closed; 2) the null subspaces of the operators B and B^* are finite-dimensional.

In what follows it is assumed that the operator $B = -\partial F(x_0, y_0)/\partial y$ (Fréchet derivative) is an F-operator, and that the dimensions of the null subspaces of the operators B and B^* are respectively n and m, $n + m > 0$. The number $\chi = n - m$ will be called the index of the operator B.

Let $F(x, y)$ be continuously differentiable in the Fréchet sense the required number of times in some neighborhood of the point (x_0, y_0). Put $x - x_0 = h$, $y - y_0 = g$; then

$$F(x, y) \equiv F(x_0 + h, y_0 + g) - F(x_0, y_0) \equiv Ah - Bg + L(h, g),$$

$$A = \frac{\partial F(x_0, y_0)}{\partial x}, \qquad B = -\frac{\partial F(x_0, y_0)}{\partial y}, \qquad L(h, g) = O(\|h\|^2 + \|g\|^2).$$

Write equation (1) as

$$Bg = Ah + L(h, g). \tag{1’}$$

Here, of course, $g(0) = 0$.

Let $\{\varphi_i\}_1^n$ be a basis of the null subspace of the operator B; $\{\psi_i\}_1^m$ a basis of the null subspace of the operator B^*; $\{\gamma_i\}_1^n$ a system of elements from E_1^* and $\{v_i\}_1^m$ is a system of elements from E_2, chosen so that

$$\gamma_r(\varphi_s) = \delta_{rs}, \quad r, s = 1, 2, \dots, n; \qquad \psi_k(z_l) = \delta_{kl}, \quad k, l = 1, 2, \dots, m.$$

Introduce the projection operators

$$P = \sum_{i=1}^{m} \psi_i(\cdot) z_i, \qquad Q = \sum_{j=1}^{n} \gamma_j(\cdot) \varphi_j.$$

Obviously, $P^2 = P$; $Q^2 = Q$; $BQ = PB = 0$. The operators P and Q generate the following direct decompositions of the spaces E_1 and E_2:

$$E_1 = E_1^{\infty - n} + E_1^n; \qquad E_2 = E_{2,\infty-m} + E_{2,m}, \tag{2}$$

where $E_1^n = QE_1$; $E_1^{\infty-n} = (J - Q)E_1$; $E_{2,m} = PE_2$; $E_{2,\infty-m} = (J - P)E_2 = BE_1$.

We proceed to the study of equation $(1')$.

Case 1. $n > 0$, $m = 0$.

Here $E_{2,m} = 0$. Setting $g = u + v$, where $u \in E_1^{\infty-n}$, $v \in E_1^n$, we have

$$\widetilde{B} u = Ah + L(h, u + v),$$

where $\widetilde{B}$ is equal to B on $E_1^{\infty-n}$ and is invertible.

By the Hildebrandt-Graves theorem on implicit operators, for any sufficiently small h and v this equation has a unique solution $u(h, v)$ such that $u(0, 0) = 0$.

Consequently, every solution of equation (1) has the form

$$y = y_0 + u(h, v) + v.$$

As v one may take an arbitrary operator of h with range in E_1^n, provided only that $v(0) = 0$.

Case 2. $n = 0,\ m > 0$.

Here $E_1^n = 0$. Applying successively the operators $J - P$ and P to $(1')$, we have

$$Bg = Ah + L(h, g),$$

$$0 = PAh + PL(h, g).$$

From the first equation we find $g(h)$ for all sufficiently small h and substitute it into the second equation. We obtain: for sufficiently small h from the nonlinear manifold

$$PAh + PL(h, g(h)) = 0$$

(which may be empty), equation (1) has a unique solution that becomes y_0 when $x = x_0$.

Case 3. $m > 0,\ n > 0$.

Setting $g = u + v$, where $v \in E_1^n,\ u \in E_1^{\infty - n}$, and applying the operators $J - P,\ P$ to $(1')$, we obtain

$$\widetilde{B}u = Ah + L(h, u + v),$$

$$0 = PAh + PL(h, u + v). \tag{1”}$$

Here $u(0) = v(0) = 0$. The operator $\widetilde{B}$ is equal to B on $E_1^{\infty - n}$ and is invertible. Equations (1), $(1')$, and $(1'')$ are obviously equivalent.

For sufficiently small h and v, the first equation has a unique solution $u(h, v)$ such that $u(0, 0) = 0$. Substituting it into the second equation, we have

$$0 = PAh + PL(h, u(h, v) + v).$$

This is the Lyapunov-Schmidt branching equation. It can be written out as a system of m numerical equations with n unknowns.

The results obtained make it possible to apply the methods of branching theory to nonlinear singular integral equations. In partic-

···of continuity, V. K. Natalevich’ s theorems (^{8}), established by him for a nonlinear singular equation with a Hilbert-type kernel, follow directly from what was set forth above.

Below, in order to simplify the exposition, we assume that $m = n$ $(\chi = 0)$.

We shall seek solutions of system $(1'')$ representable in the form

$$u = u_\varepsilon + U, \qquad v = v_\varepsilon + V, \tag{3}$$

where u_ε and v_ε are homogeneous operators of order $\varepsilon > 0$ in h; $\|U(th)\| = o(t^\varepsilon)$, $\|V(th)\| = o(t^\varepsilon)$ as $t \to 0$.

Lemma. Suppose there exists a solution of the form (3) of system $(1'')$. Then: (1) if ε is not a natural number, then $u_\varepsilon = 0$; 2) if $\varepsilon = k$, where k is a natural number, then $u_\varepsilon = \widetilde{B}^{-1} F_{k0} h^k$ (everywhere below, for brevity, we put

$$F_{rs} = \frac{1}{(r+s)!}\frac{\partial^{r+s} F(x_0, y_0)}{\partial x^r \partial y^s}, \quad r + s \geq r, \qquad F_{10} = A.$$

The proof of the lemma follows from the uniqueness of the expansion in homogeneous operators. The lemma makes it possible to find the leading term of the solution without forming the branching equation.

Let now $L(h, g)$ be expanded in an absolutely convergent F-power series in (h, g) in a neighborhood of the point $(0, 0)$. Fix $h = h^0$ and consider $(1'')$ on the ray th^0, $t > 0$.

Write the second of equations $(1'')$ in the form:

$$0 = \sum_{\alpha=0}^{\infty} \left(\sum_{\beta=\rho_\alpha}^{\infty} PF_{\beta\alpha} h^\beta \right) (u + v)^\alpha, \tag{4}$$

where ρ_α is such that $PF_{i\alpha} h^{0i} = 0,\ i = 0, 1, \dots, \rho_\alpha - 1$, and $PF_{\rho_\alpha \alpha} h^{0\rho_\alpha} \neq 0$.

Now, to determine ε, one may use the Newton diagram (^{3}). Having found ε, we choose v_ε (u_ε is known, see the lemma) so that the terms of lowest order in h in (4) vanish. Suppose that on the diagram one of the found values $\varepsilon > 0$ corresponds to a segment with endpoints (α, ρ_α) and (β, ρ_β), $\beta \geq \alpha$. In order that the terms of the lowest order in h (denote this order by σ) cancel in (4), it is necessary and sufficient that

$$\sum_{i=\alpha}^{\beta} PF_{\rho_i i} h^{\rho_i} (u_\varepsilon + v_\varepsilon)^i = 0. \tag{5}$$

This equation can be written as a system of n equations of order β with n unknowns. Suppose it is solvable. We restrict ourselves to the case of simple roots $\bar{v}_\varepsilon$ of equation (5), when the operator

$$B_{\sigma-\varepsilon}(h) = \sum_{i=\alpha}^{\beta} iPF_{\rho_i i} h^{\rho_i} (u_\varepsilon + \bar{v}_\varepsilon)^{i-1}$$

is invertible. Then necessarily ε is a rational number and $\beta > \alpha$. For U and V we have the system

$$\widetilde{B}U = R_1(h, u_\varepsilon + \bar{v}_\varepsilon + U + V), \tag{6}$$

$$C_{\sigma-\varepsilon}(h)U + B_{\sigma-\varepsilon}(h)V = R_2(h, u_\varepsilon + \bar{v}_\varepsilon, U + V),$$

where

$$R_1(th, t^\varepsilon g) = o(t^\varepsilon), \qquad R_2(th, t^\varepsilon g_\varepsilon, t^\varepsilon G) = o(t^\sigma) \quad \text{as } t \to 0.$$

To (6) one can apply the arguments carried out in (^{9}) for the particular case $\varepsilon = 1,\ \sigma = 2$ (see also (^{10}), where $\varepsilon = \frac{1}{2},\ \sigma = 1$).

We formulate the results.

Theorem 1. Suppose: 1) $\varepsilon > 0,\ \sigma > 0$ have been found by means of the Newton diagram; 2) $F(x, y)$ is $1 + \max_{\rho_i + i\varepsilon = \sigma}(\rho_i, i)$ times continuously differentiable in the Fréchet sense in a neighborhood of the point (x_0, y_0); 3) $\partial F(x_0, y_0)/\partial y$ is an F-operator with zero index ($n \geqslant 1$); 4) equation (5) is nontrivially solvable and has N simple roots in the c-star Ω.

Then there exists a nonempty c-star Ω', in which equation (1) has N solutions of the form

$$y = y_0 + g_\varepsilon(x - x_0) + G(x - x_0).$$

Theorem 2. If, under the conditions of Theorem 1, $F(x, y)$ expands into an absolutely convergent F-power series in a neighborhood of the point (x_0, y_0), then each of the solutions expands into an absolutely convergent series in homogeneous operators (10)

$$y = y_0 + \sum_{i=0}^{\infty} g_{\varepsilon + i/q}(x - x_0),$$

where q is the denominator of the number ε.

Remark 1. The same method is applicable also in the case of multiple roots of equation (5), and also for $\chi \neq 0$, but this leads to considerable complications.

Remark 2. Finding, with the aid of the Newton diagram, values $\varepsilon < 0$ gives special solutions of equation (1).

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Note: Figure translations are in progress. See original paper for figures.

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