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Abstract

Full Text

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CYBERNETICS AND CONTROL THEORY

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ON OPTIMAL CONTROL OF SYSTEMS WITH DISTRIBUTED PARAMETERS

(Presented by Academician V. S. Kulebak'in on 17 III 1960)

In the theory of optimal control developed in recent years, control problems have been considered only for objects with lumped parameters, described by ordinary differential equations. The solution of these problems may be based on L. S. Pontryagin's maximum principle^(3-7,11), on Bellman's dynamic programming method⁽⁸⁾, or on the use of isosurfaces in the phase space of the system⁽⁹⁻¹⁰⁾. However, in many technical problems one encounters controlled objects whose parameters are distributed in space, and the distribution functions and their boundary values play an essential role in the control processes.

Many controlled objects are described by a system of first-order partial differential equations

$$f_i \left(x, t, Q, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial t}, u, v, w \right) = 0, \quad i = 1, 2, \dots, n, \quad (1)$$

where x and t are independent variables from the domain $l_0 \leq x \leq l_1$, $t_0 \leq t \leq t_1$, and $Q = Q(x, t) = (Q_1(x, t), \dots, Q_n(x, t))$ is a vector function characterizing the state of the object; $u = u(t) = (u_1(t), \dots, u_k(t))$ is a controlling vector function of one variable t ; $v = v(x, t) = (v_1(x, t), \dots, v_r(x, t))$ is a controlling vector function of two independent variables x and t ; $w = w(x) = (w_1(x), \dots, w_s(x))$ is a controlling vector function of one independent variable x .

In this case constraints may be imposed: all or part of the partial derivatives up to order p with respect to the coordinates of the controlling vector functions may be bounded above and below by certain quantities. For example, there may be constraints of the following kind:

$$\left| \frac{d^\alpha u}{dt^\alpha} \right| \leq K, \quad \left| \frac{d^\beta w}{dx^\beta} \right| \leq N, \quad \left| \frac{\partial^{\gamma+\delta} v}{\partial x^\gamma \partial t^\delta} \right| \leq M, \quad (2)$$

where K, N, M are certain specified constants.

Three optimal-control problems may be formulated:

1. To vary the control actions $u(t), v(x, t), w(x)$, subject to the prescribed constraints, so that, under the initial condition

$$Q(x, t_0) = Q_0(x) \quad (3)$$

and the boundary condition

$$Q(l_0, t) = Q_b(t) \quad (4)$$

the functional

$$I = I \left(x, t, Q, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial t}, u, v, w \right) \quad (5)$$

attains its minimum.

2. Vary the control actions $u(t), v(x, t), w(x)$ in such a way that the functional of type (5), for fixed values $x \in [l_0, l_1]$, attains its minimum.
3. Vary the control actions $u(t), v(x, t), w(x)$ in such a way that the functional of type (5) attains its minimum for fixed values $t \in [t_0, t_1]$.

In addition, an extra condition may be imposed: that the function $Q(x, t_1)$ or $Q(l_1, t)$ lie in some prescribed ε -neighborhood of a given vector function $Q^*(x)$ or $Q^*(t)$; for example, it is necessary that, for $t = t_1$, the condition

$$\max_{x \in [l_0, l_1], i=1, \dots, n} |Q_i^*(x) - Q_i(x, t_1)| \leq \varepsilon. \quad (6)$$

be satisfied.

Problems of obtaining a distribution $Q(x, t_1)$ close to a prescribed one in the shortest time $t_1 - t_0$ belong to this type of problem.

Furthermore, analogous problems can be posed for partial differential equations of order higher than the first.

Let, for example, an equation of the type of the heat-conduction equation be given,

$$\frac{\partial Q}{\partial t} = a \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) + v_1 \frac{\partial Q}{\partial x} + v_2 \frac{\partial Q}{\partial y} + v_3 \frac{\partial Q}{\partial z} + g(P, t, u_1, \dots, u_k) \quad (7)$$

with the initial condition

$$Q(P, 0) = Q_0(P) \quad (8)$$

and the boundary condition

$$\left. \frac{\partial Q}{\partial n} \right|_{\Gamma} + \mu Q|_{\Gamma} = \psi(P, t, u_{k+1}, \dots, u_r)|_{\Gamma}. \quad (9)$$

Here $P = P(x, y, z)$ is a point of the body D , bounded by the surface Γ , in which the process under consideration takes place; n is the outward normal; a, v_1, v_2, v_3, μ are prescribed functions of P and t ; g and ψ are prescribed functions of their arguments. The control actions $u_1(t), \dots, u_r(t)$ may be subject to the restrictions

$$\left| \frac{d^\alpha u_j}{dt^\alpha} \right| \leq K_j, \quad j = 1, 2, \dots, r, \quad (10)$$

where all K_j are prescribed constants.

The problem is posed as follows: vary the control actions $u_1(t), \dots, u_r(t)$, subject to condition (10), so that the functional

$$I = I(P, t, Q, u_1, \dots, u_r) \quad (11)$$

attains its minimum. In this case an additional restriction may be imposed: the function $Q(P, t_1)$ must lie in some ε -neighborhood of a given function $Q^*(P)$.

In addition, the following problem is of interest: vary $u_1(t), \dots, u_r(t)$, subject to condition (10), so that the functional (11) attains a minimum at a point $P \in \Gamma' \subset \Gamma$, where Γ' is some prescribed part of the boundary Γ . We note that here, as in the case of a first-order system, the control functions may depend not only on t , but also on x , on x and y , and also on x, y, z .

As an example, we consider the problem of optimal control of a continuous furnace, in which "thin" workpieces ($Bi \leq 0.25$) are heated while moving from the furnace entrance to the furnace exit.

The heating equation has the form

$$av \frac{\partial x}{\partial Q} + a \frac{\partial Q}{\partial t} + Q - u = 0, \quad 0 \leq x \leq L, \quad 0 \leq t \leq T. \quad (12)$$

Initial condition

$$Q(x, 0) = Q_0(x), \quad 0 \leq x \leq L. \quad (13)$$

Boundary condition

$$Q(0, t) = 0, \quad 0 \leq t \leq T. \quad (14)$$

Constraint conditions

$$u_1 \leq u(t) \leq u_2, \quad 0 \leq t \leq T. \quad (15)$$

Here $Q = Q(x, t)$ is the temperature of the metal at the point x at time t ; $v = v(t) \geq 0$ is the speed of motion of the metal along the furnace; a is a coefficient characterizing the thermal properties of the metal, depending on the difference

$$x - \int_0^t v(\rho) d\rho;$$

$u = u(t)$ is the temperature of the working space of the furnace, serving as the control action; u_1, u_2 are prescribed quantities; L is the length of the furnace; T is the prescribed time; $Q_0(x)$ is a prescribed function.

Let us note that equation (12) describes the operation of heat-exchange apparatuses and other units. This equation can also be regarded as an equation for increments. The optimal-control problem consists in varying $u = u(t)$, subject to condition (15), in such a way that the functional

$$I = \int_0^T [Q_3 - Q(L, t)]^2 dt \quad (16)$$

attains a minimum; Q_3 is a prescribed constant.

This is a problem of type 2, which it has been possible to reduce to a problem solved by means of L. S. Pontryagin's maximum principle. The optimal control has the form

$$u = u(t) = \begin{cases} u_2, & \text{if } \psi_1 > 0, \\ u_1, & \text{if } \psi_1 < 0, \\ Q_3 + av(t)e^{-t/a}Q'_0 \left(L - \int_0^t v(\rho) d\rho \right), & \text{if } \psi_1 = 0. \end{cases}$$

Here

$$\psi_1(t) = e^{-t/a} \left\{ C + \int_0^t \left[Q_3 - e^{-\tau/a} Q_0 \left(L - \int_0^\tau v(\rho) d\rho \right) - \Delta(\tau) \right] e^{\tau/a} d\tau \right\};$$

$\Delta(t)$ satisfies the equation

$$a\Delta'(t) + \Delta(t) = u(t) \quad \text{with } \Delta(0) = 0$$

and C is chosen from the condition that $\psi_1(T) = 0$.

On the basis of these relations one can synthesize an optimal-control system. As is seen from formula (17), in order to form the control action it is necessary for the control system to contain a model of the object in the event that direct measurements are impossible.

Conclusions. 1. In automating objects with distributed parameters, there arises the need for optimal control under constraints imposed on the control actions.

2. A characteristic feature is that, in addition to concentrated control actions, there are control actions distributed in space.
3. It is essential that the control actions may enter both into the basic equations of the process and into the boundary conditions.
4. In connection with the problem of attaining, in some sense, a minimal deviation from a prescribed distribution, the question of the attainability of the prescribed state is of substantial importance.
5. The known works on the theory of optimal control do not directly provide a more or less general method for solving the problems posed. However, in some cases a problem of type (12) can be reduced to a problem solvable by means of L. S. Pontryagin's maximum principle.
6. The optimal control of objects with distributed parameters is a function of the distribution of the states of these objects, which leads to the necessity of modeling this distribution by means of devices with a sufficiently large operative memory.

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