LACUNAS OF NONHYPERBOLIC EQUATIONS

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Abstract

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MATHEMATICS

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LACUNAS OF NONHYPERBOLIC EQUATIONS

(Presented by Academician I. G. Petrovskii on 25 II 1960)

It is well known that for hyperbolic equations the domain of dependence of solutions of the Cauchy problem is finite; this means that the solution at a given point $M(t, x_1, \dots, x_n)$ does not depend on the values of the initial data on the plane $t = 0$ outside the base of the characteristic cone for the point M. In what follows we shall assume that the equation under consideration is linear and has constant coefficients.

Domains of the plane $t = 0$ possessing the property that the values of the initial data on them do not affect the value of the solution at the point M are called lacunas. Thus the region exterior to the base of the characteristic cone is always a lacuna for a hyperbolic equation.

I. G. Petrovskii, in his well-known work (^{1}), established that in certain cases there may be lacunas also inside the base of the characteristic cone. He established necessary and sufficient conditions for the existence of lacunas in terms of the topological structure of the algebraic surface that is the base of the cone (directrix) reciprocal to the characteristic (the cone of normals). V. A. Borovikov (^{2}) obtained a number of sufficient criteria both for the existence and for the absence of lacunas of a hyperbolic equation, without appealing to the topological structure of the mentioned algebraic surface.

F. John (^{3}) proved that if an equation, solved with respect to the derivative of order l with respect to t, where l is the order of the equation, has a finite domain of dependence (i.e., a neighborhood of infinity belongs to a lacuna), then the equation is hyperbolic (in the broad sense), or the operator decomposes into factors, one of which is hyperbolic.

As far as I know, up to now there have been no examples of nonhyperbolic equations having lacunas.

1. In the present paper a class of equations is established which are not hyperbolic and which have lacunas.

The simplest equation of this type is the equation:

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$\frac{\partial^2 \Delta u}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial^4 u}{\partial x_i^4}. \tag{1}$$

The equation of the directrix cone reciprocal to the characteristic one is written as follows:

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{n} \xi_i^4. \tag{2}$$

In the case $n = 2$ the "surface" has the form shown in Fig. 1; Fig. 2 gives an image of the reciprocal "surface," i.e., the base of the characteristic cone. It is easy to imagine the form of the surface also for $n > 2$.

Applying the results of our investigation to equation (1), we obtain:

Theorem. Construct the convex hull of the surface (2) and take the figure H_0 (Fig. 2) reciprocal to this convex hull. Then, if $n > 3$ and is odd, the domain H_0 is a lacuna.

2. Consider the equation

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) u = 0, \tag{3}$$

where $Q(\lambda, \xi_1, \dots, \xi_n)$ is a homogeneous polynomial with constant coefficients of degree l with respect to $\lambda, \xi_1, \dots, \xi_n$ and of degree $m < l$ with respect to λ. Expanding this polynomial in powers of λ, we obtain:

$$Q(\lambda, \xi_1, \dots, \xi_n) = P_{l-m}(\xi_1, \dots, \xi_n)\lambda^m + \dots + P_l(\xi_1, \dots, \xi_n), \tag{4}$$

where $P_k(\xi_1, \dots, \xi_n)$ is a homogeneous polynomial of degree k with respect to $(\xi_1, \dots, \xi_n)$. By ξ and x we shall denote the vectors $\xi = (\xi_1, \dots, \xi_n)$ and $x = (x_1, \dots, x_n)$.

Fig. 1

Fig. 2

We shall assume that:

a) the polynomial $P_{l-m}(\xi)$ does not vanish on the sphere $|\xi|^2 = 1$; consequently $l - m$ is even;

b) the polynomial $P_l(\xi)$ also does not vanish on the sphere $|\xi|^2 = 1$; consequently l is even;

c) the equation

$$Q(\lambda, \xi) = 0 \tag{5}$$

has, with respect to λ, exactly m real and distinct roots on the sphere $|\xi|^2 = 1$.

These roots $\lambda_1 = \lambda_1(\xi), \dots, \lambda_m = \lambda_m(\xi)$ will be continuous (even analytic) functions of $\xi = (\xi_1, \dots, \xi_m)$ on the sphere $|\xi|^2 = 1$; of them $m/2$ will be positive and $m/2$ negative.

We note that $Q'_\lambda(\lambda, \xi)\big|_{|\xi|^2=1} \neq 0$, since all roots are distinct; hence also $\operatorname{grad} Q(1, \xi) \neq 0$ for $|\xi| \neq 0$.

Thus, the surface

$$Q(1, \xi) = 0, \tag{6}$$

which is the director surface of the cone (5), consists of $m/2$ closed surfaces—ovals—having no singular points; these ovals are nested one inside another.

Moreover, the coordinate origin $\xi = 0$ is an $(l - m)$-fold point of the surface and lies inside the smallest oval.

For equations (3) satisfying only the conditions just listed, formulas are established for the fundamental solution of the Cauchy problem, analogous to the well-known formulas of Herglotz-Petrovskii.

Recall that the solution $K(x - y, t)$ is a fundamental solution of the Cauchy problem if it satisfies the initial conditions:

$$\frac{\partial^s K(x - y, 0)}{\partial t^s} = 0 \quad \text{for } s = 0, 1, \dots, m - 2;$$

$$\frac{\partial^{(m-1)} K(x - y, 0)}{\partial t^{m-1}} = \delta(x - y),$$

where $\delta(x) = \delta(x_1, \dots, x_n)$ is the δ-function.

The formulas for the fundamental solutions are as follows:

$$K(x - y, t) = a_{n,m} \int_{Q(1,\xi)=0} \frac{(-1)^N \omega\big[(x - y)\xi + t\big]}{|\operatorname{grad} Q(1, \xi)|} P_{l-m}(\xi)\, ds. \tag{7}$$

Here:

1) The integration is carried out over all $m/2$ ovals of the surface $Q(1, \xi) = 0$.
2) N is the ordinal number of the oval, counted from the origin.
3) The function $\omega(s)$, ordinary or generalized in the sense of Gelfand-Shilov (^{4}), is determined by the following equalities:

a) for odd n

$$\omega(s) = s^{m-n-1} \operatorname{sign} s, \qquad \text{if } m - 1 \geq n,$$

$$\omega(s) = \delta^{(n-m)}(s), \qquad \text{if } n > m - 1;$$

b) for even n

$$\omega(s) = s^{m-n-1} \ln \frac{s}{s-t}, \qquad \text{if } m - 1 \geq n,$$

$$\omega(s) = \frac{1}{s^{n-m+1}}, \qquad \text{if } n > m - 1.$$

4) $a_{n,m}$ is a constant for given n and m.
5) ds is the surface-area element.

We note that if $l = m$, equation (3) becomes hyperbolic, $P_0(\xi) = \text{const}$, and formulas (7) will be the formulas of Gerglotz-Petrovskii.

Theorem 1. *Under the conditions listed above for equation (3), if n is odd and $n > m - 1$, then the connected component of the base of the characteristic cone in the plane $t = 0$, containing the projection of the vertex of the cone, is a lacuna.*

Denote this component by H_0. From consideration of formula (7) for odd n and for $n > m - 1$, it is easy to establish that $K(x - y, t)$ vanishes in the region H_0, and this means that H_0 is a lacuna.

Remark. Using formulas (7), one can also obtain certain other criteria for the existence or absence of lacunae for equation (3), for example the criteria of V. A. Borovikov (^{2}).

3. Consider the equation

$$Q_\varepsilon \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) u = 0, \tag{8}$$

where

$$Q_\varepsilon(\lambda, \xi) = a_{\frac{l-m}{2}} \varepsilon^{l-m} \lambda^l + \dots + a_2 \varepsilon^4 \lambda^{m+4} |\xi|^{l-m-4} +$$

$$+ a_1 \varepsilon^2 \lambda^{m+2} |\xi|^{l-m-2} + Q(\lambda, \xi).$$

It is always possible to choose numbers a_i so that, for all sufficiently small values of ε, the polynomial $Q_\varepsilon(\lambda, \xi)$ corresponds to a strictly hyperbolic equation.

Let

$$a_{\frac{l-m}{2}} \varepsilon^{l-m} K_\varepsilon(x - y, t)$$

be the fundamental solution of the Cauchy problem for the hyperbolic equation (8).

Denote

$$K_\varepsilon^*(x - y, t) = P_{l-m}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) K_\varepsilon(x - y, t).$$

Of course, both $K_\varepsilon(x, t)$ and $K_\varepsilon^*(x, t)$ may be either ordinary or generalized functions. Then the following holds.

Theorem 2.

$$\lim_{\varepsilon \to 0} K_\varepsilon^*(x, t) = K(x, t),$$

where $K(x - y, t)$ is the fundamental solution of the Cauchy problem for equation (3).

Therefore, if in a neighborhood of some point $(x_1, \dots, x_n)$, for all sufficiently small ε, the function $K_\varepsilon^*(x, t) = 0$, then also $K(x, t) = 0$. Consequently, this neighborhood belongs to a lacuna of equation (3). This makes it possible to obtain criteria for the existence of lacunae for equation (3) from the corresponding criteria established for the hyperbolic case.

Thus, equation (3) may be regarded as the limiting equation for the hyperbolic equation (8).

As $\varepsilon \to 0$, the interiors of the $\frac{l-m}{2}$ ovals of the surface $Q_\varepsilon(1, \xi) = 0$ contract to a point, while the remaining $\frac{m}{2}$ ovals turn into the ovals of surface (5).

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Note: Figure translations are in progress. See original paper for figures.

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