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## Abstract

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## MATHEMATICAL PHYSICS

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# MAJORIZATION OF FEYNMAN DIAGRAMS

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To establish dispersion relations in perturbation theory, a method of majorizing Feynman diagrams was proposed in works (1–2). In the present note we present the results of a consistent development of the idea of majorizing diagrams. For illustration, we give here an application of the developed method to the vertex part.

To each Feynman diagram $D$ there corresponds a quadratic form $Q_D$ in the external momenta $p_a$. It is defined as follows. Let the 4-momenta $k_\nu$ be assigned on the internal lines of the diagram in such a way that at each of its vertices the law of conservation of momentum is satisfied. Then the $k_\nu$ are linear functions of the external momenta $p_a$ and of independent "internal" momenta $t_i$. Let, further,

$$K_D(\alpha, p, t) = \sum_{\nu=1}^{l} \alpha_\nu \left(k_\nu^2 - m_\nu^2\right) = \sum_{i,j} a_{ij} t_{it} j - 2 \sum_i b_i t_i + c, \tag{1}$$

where $l$ is the number of internal lines of the diagram, $\alpha_\nu \geqslant 0$. Then the quadratic form $Q_D$ is determined by the equality

$$Q_D(\alpha, p) = \frac{\begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix}}{|a_{ij}|}. \tag{2}$$

Expression (2) is obtained from (1) if, instead of $t$, one substitutes the solution of the system of linear equations

$$\frac{1}{2} \frac{\partial K_D}{\partial t_i} \equiv \sum_j a_{ij} t_j - b_i = 0. \tag{3}$$

In what follows we shall consider such external momenta whose scalar products are real. Define the domain of variation of the set of external momenta $G_\varepsilon(D)$ by the inequality

$$Q_D(\alpha, p) < -\varepsilon \sum_{\nu=1}^{l} \alpha_\nu \quad \text{for all } \alpha_\nu \geqslant 0, \qquad \sum_{\nu=1}^{l} \alpha_\nu > 0. \tag{4}$$

In the sum of domains $G(D) = \bigcup_{\varepsilon>0} G_\varepsilon(D)$, the integral $T_D$, representing the regularized matrix element, has no singularities (3).

Let $R$ be some set of connected diagrams of a definite process.* In the intersection of the domains $G_R = \bigcap_{D\in R} G(D)$, each integral $T_D$, $D \in R$, has no singularities. If, for two diagrams $D_1 \in R$ and $D_2 \in R$, it is known that $G(D_1) \subseteq G(D_2)$, then in finding $G_R$ from the two diagrams $D_1, D_2$ it suffices to take into account only the diagram $D_1$. In this case we shall say that the diagram $D_1$ majorizes the diagram $D_2$, or that the diagram $D_2$ is majorized by the diagram $D_1$, and denote this by $D_1 < D_2$, or $D_2 > D_1$.

Let $P$ denote the set of all vectors of the form $p = \sum_a A_a p_a$, where $A_a$ are real numbers and $p_a$ are the external momenta. In what follows we shall not consider the general case of real scalar products of external momenta, but shall restrict ourselves to the special case when*

$$p^2 = \left(\sum_a A_a p_a\right)^2 \geqslant 0 \tag{5}$$

for arbitrary real $A_a$. In this case it follows directly from (1), (2), and (3) that**:

**Lemma.** *The form $Q_D$ is equal to the smallest value of the form $K_D$ under the condition that the vectors $k_\nu$ satisfy the law of conservation of momentum at each vertex of the diagram and take values in $P$.*

The following two theorems play the main role in the majorization of diagrams.

In order to formulate the first theorem, let us define the notion of a subdiagram. Namely: if, as a result of deleting from a diagram $D \in R$ some internal lines and internal vertices***, a diagram $D' \in R$ is obtained, then the diagram $D'$ will be called a subdiagram of the diagram $D$ (relative to $R$).

**Theorem 1.** *Every diagram is majorized by any of its subdiagrams.*

**Proof.** Let $k_\nu$ be the momenta on the internal lines of the subdiagram. On each internal line of the diagram subject to deletion we put the momentum equal to zero. Then the law of conservation of momentum will be satisfied at each vertex of the diagram. Suppose several lines $L_{rs}$, $s = 1, \dots, 1 + n_r$, $n_r \geqslant 0$, of the diagram are combined into one line $L_r$ of the subdiagram. In this case

$$k_{rs} = k_r, \qquad m_{rs} = m_r, \qquad s = 1, \dots, 1 + n_r.$$

* For example, the set of all strongly connected diagrams.

Let $\alpha_{rs}$ denote the Feynman parameter of the line $L_{rs}$, and $\beta_\nu$ the parameters of the deleted lines. Then

$$K_D = -\sum_\nu \beta_\nu m_\nu^2 + \sum_r \alpha_r \left(k_r^2 - m_r^2\right), \qquad \text{where } \alpha_r = \sum_{s=1}^{1+n_r} \alpha_{rs}. \tag{6}$$

Hence, by virtue of the lemma, the assertion of the theorem follows.

**Theorem 2.** *Suppose the diagram $D$ contains a closed $(n+1)$-gon, to $n$ sides of which there corresponds a mass $M$, and to one side a mass $m \leqslant M$. Change the masses on these sides in the following way: $M \to m$, $m \to M$. As a result we obtain a new diagram $D'$, and moreover****,*

$$G(D') \subseteq G(D).$$

**Proof.** Let $k_1, \dots, k_n, k_{n+1} \in P$ be the momenta on the sides of the $(n+1)$-gon. If to each of these momenta we add an arbitrary momentum $t$, while leaving the remaining momenta of the diagram unchanged, then the law of conservation of momentum at each vertex of the diagram will not be violated. The least value of the form $K_{D'}$ over $t \in P$ is equal to

$$\overline{K}_{D'} = \frac{\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}\alpha_i'\alpha_j'(k_i-k_j)^2}{\alpha_1'+\dots+\alpha_{n+1}'} - m^2(\alpha_1'+\dots+\alpha_n') - M^2\alpha_{n+1}' + \sum_\nu \beta_\nu(q_\nu^2 - m_\nu^2), \tag{7}$$

where $\alpha_i'$ are the Feynman parameters of the sides of the $(n+1)$-gon, and $\beta_\nu$ are the parameters of the remaining lines of the diagram. If one sets

$$\alpha_i' = \frac{\varkappa\alpha_i}{m^2}, \quad i = 1, \dots, n; \qquad \alpha_{n+1}' = \frac{\varkappa\alpha_{n+1}}{M^2}; \qquad \varkappa = \frac{M^2(\alpha_1+\dots+\alpha_n) + m^2\alpha_{n+1}}{\alpha_1+\dots+\alpha_{n+1}}, \tag{8}$$

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* Such an assumption in papers ($^{1,2,4,5}$) is called the Euclidean condition.

** From this lemma there follows directly the assertion of Nakanishi' s main theorem (Theorem 2 of paper ($^5$)).

*** An internal vertex of a diagram is a vertex at which the external momentum is equal to zero. Otherwise the vertex is called external.

**** If the diagram $D$ belongs to $R$, then the diagram $D'$, generally speaking, does not belong to $R$; however, its subdiagram may belong to $R$. In this case such a subdiagram majorizes the diagram $D$.

then

$$\overline{K}_{D'} = \overline{K}_D + \frac{M^2 - m^2}{m^2} \frac{\sum_{i=2}^{n} \sum_{j=1}^{i-1} \alpha_i \alpha_j (k_i - k_j)^2}{\alpha_1 + \cdots + \alpha_{n+1}}. \tag{9}$$

Hence, by the lemma, it follows that

$$G(D') \subseteq G(D).$$

As an example, let us consider the set $R$ of diagrams defined as follows. Every $D \in R$ is a strongly connected diagram* of the $\pi$-meson–nucleon vertex part. At each vertex of a diagram $D \in R$ there meet three and only three lines: an even number (2 or 0) of baryon lines and an odd number (1 or 3) of meson lines. The diagram $D \in R$ takes into account only "strong" interactions, so that the $\pi$-meson mass is the smallest in $D$, while the nucleon mass is the smallest among the masses corresponding to lines carrying baryon charge. The consideration of only strongly connected diagrams is due to the fact that in this case $G_R$ gives the boundary of the continuous spectrum.

By virtue of the law of conservation of baryon charge, in each diagram $D \in R$ there is one unclosed broken line (polygon) formed by the lines carrying baryon charge. Upon replacing all the lines of this polygon by nucleon lines, and the remaining lines of the diagram by $\pi$-meson lines, the form $Q_D$ increases. Therefore $G_R = G_{R^*}$, where $R^*$ is the subset of diagrams $R^* \subset R$ in which there is one nucleon polygon,** and all the remaining lines are $\pi$-meson lines.

Thus, in order to determine $G_R$, it is sufficient to consider the set of diagrams $R^*$.

Let $a$ and $b$ denote the external nucleon vertices, and $c$ the external meson vertex of a diagram $D \in R^*$. Two cases are possible, depending on whether the vertex $c$ is situated on the nucleon polygon or lies outside this polygon. In the second case, define the characteristic point $\tilde{a}$ of the diagram as the point of the nucleon polygon nearest to $a$ from which there exists a continuous path to the point $c$ passing only along meson lines. In an analogous way define the characteristic point $\tilde{b}$. The points $\tilde{a}$ and $\tilde{b}$ do not coincide. Conversely, if the point $c$ lies on the nucleon polygon, then we shall regard $c$ both as the point $\tilde{a}$ and as the point $\tilde{b}$. This makes it possible to consider both cases in a unified manner.

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* A diagram is called strongly connected if it does not split into two parts after cutting any one internal line.

** That is, a broken line formed by nucleon lines.

We shall show that $G_{R^*} = G_{R^{**}}$, where $R^{**}$ is the subset of diagrams $R^{**} \subset R^*$ in which the external points $a$ and $b$ are characteristic (in this case the point $c$ cannot lie on a nucleon polygon). Indeed, let in a diagram $D \in R^*$ the point $\tilde{a}$ not coincide with the point $a$. It is not difficult to show that in such a diagram there exists a subdiagram, partially shown in Fig. 1. Between the points $a$ and $\tilde{a}$ of this figure there are $2n-1$ nucleon lines $\lambda_1, \ldots, \lambda_{2n-1}$, $n \geqslant 1$. Let $n > 1$. Applying the operation of Theorem 2 to the triangle with sides $\lambda_1, \lambda_2, l_1$, and then removing the line $\lambda_2$ together with the vertices it contracts, we obtain a diagram of Fig. 1 with $n' = n-1$. After $n-1$ such operations we obtain the diagram of Fig. 1 with $n' = 1$. Applying the operation of Theorem 2 to the polygon of this diagram formed by the meson line $l_1$ and by the nucleon lines enclosed between the vertices $a$ and $a'$, we obtain a diagram in which the point $a$ is characteristic.

**Fig. 1**

If in the latter diagram the point $b$ is noncharacteristic, then as a result of applying the described procedure once more we arrive at a diagram in which both points $a$ and $b$ are characteristic. By Theorems 1 and 2 this diagram majorizes the original diagram $D$. Thus, $G_{R^{**}} = G_{R^*}$.

It is not difficult to show that every diagram from $R^{**}$ contains one of the two subdiagrams shown in Fig. 2. Thus, any diagram from $R^{**}$ is majorized by one of the two diagrams in this figure.

**Fig. 2**

A direct study of the quadratic forms of these diagrams shows that diagram I majorizes diagram II. Consequently,

$$G_R = G_{R^*} = G_{R^{**}} = G(I) \cap G(II) = G(I). \tag{10}$$

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*Note: Figure translations are in progress. See original paper for figures.*

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