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Abstract

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On the Application of Fractional Powers of Self-Adjoint Operators to the Study of Certain Nonlinear Differential Equations in Hilbert Space

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In the present note the nonlinear problem

$$v' + A(t, v) = 0 \quad (0 < t \leq T); \qquad v(0) = v_0 \tag{1}$$

in the Hilbert space H is studied.

To certain functions $y = y(t)$ with values in H there is associated the solution $U(t, 0; y)v_0$ of the linear problem

$$z' + A(t, y)z = 0 \quad (0 < t \leq T); \qquad z(0) = v_0,$$

and the question of the existence of a solution of problem (1) is reduced to the question of the existence of a fixed point of the operator $Uy = U(t, 0; y)v_0$.

Carrying out this scheme required a detailed analysis of the linear problem. Section 1 of the present note is devoted to this; there are given conditions for the existence of a solution of the homogeneous linear problem which substantially strengthen the result obtained in ([7]). In Section 2 problem (1) with an abstract operator is considered. The general results are applied in Section 3 to the study of parabolic equations with an elliptic operator of the second order and with nonlinear boundary conditions. We note that an important role in our considerations is played by the method of fractional powers of operators (see ([1–7])).

1. Let $\rho,\ \varepsilon_1 \in (0, 1),\ \varepsilon_2 \in (0, \rho]$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Let l be such an integer that $\rho_1 = 1 - l\rho \in (0, \rho]$. If $\rho < 1/2$, then let $\nu = \max\{\varepsilon_1 - 1 + \rho,\ \varepsilon_2 - 1 + \rho + \delta\}$, where δ is some number from $(l\rho - \rho - \varepsilon_2,\ 1 - \rho)$. If $\rho \geq 1/2$, then let $\nu = \varepsilon_1 - 1 + \rho$. Finally, let $0 \leq t \leq t + \Delta t \leq T$.

Denote by $A(t)$ $(0 \leq t \leq T)$ a positive-definite self-adjoint operator whose power ρ has a domain of definition independent of t. If $\rho \geq 1/2$, let*

$$\|A^{\rho}(t+\Delta t)A^{-\rho}(t) - A^{-\rho_1}(t+\Delta t)A^{\rho_1}(t)\| \le K_1 \Delta t^{\varepsilon_1}.$$

If, however, $\rho < 1/2$, let

$$\|A^{\rho}(t+\Delta t)A^{-\rho}(t) - I\| \le K_1 \min\{\Delta t^{\varepsilon_1},\ \Delta t^{\varepsilon_2+\delta t-\delta}\}.$$

Theorem 1. *There exists an operator $U(t,\tau)$, strongly continuous jointly in t and τ for $0 \le \tau \le t \le T$, satisfying the initial condition $U(\tau,\tau) = I$ and, for $t > \tau$, the equation $U'_t + A(t)U = 0$ in the following sense:*

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* Here and below, a bar above denotes the closure of an operator in H.

If $\nu > 0$, then for any $\alpha \in [0,\nu]$ the operator $A^{\alpha}(0)U(t,\tau)$ is continuously differentiable and

$$[A^{\alpha}(0)U(t,\tau)]'_t = -A^{\alpha}(0)A(t)U(t,\tau).$$

If, however, $\nu \le 0$, then for any $\beta \in (-\nu,\rho]$ the operator $A^{-\beta}(0)U(t,\tau)$ is continuously differentiable and

$$[A^{-\beta}(0)U(t,\tau)]'_t = -A^{-\beta}(0)A^{\beta}(t)A^{1-\beta}(t)U(t,\tau).$$

With respect to the variable τ, the operator $U(t,\tau)$ satisfies in the same sense the adjoint equation $U'_\tau - UA(\tau) = 0$.

For any $0 \le \alpha \le \beta \le \rho$ the inequalities

$$\max\{\|A^{\gamma}(t)U(t,\tau)A^{-\alpha}(\tau)\|,\ \|A^{-\alpha}(t)U(t,\tau)A^{\gamma}(\tau)\|\} \le K(\alpha,\gamma)|t-\tau|^{\alpha-\gamma};$$

$$\gamma \in [\alpha, \rho+\varepsilon_1), \quad \text{if } \rho \ge \tfrac{1}{2}; \qquad \gamma \in [\alpha, \rho+\delta+\varepsilon), \quad \text{if } \rho < \tfrac{1}{2};$$

$$\|A^{\alpha}(s)[U(t+\Delta t,\tau) - U(t,\tau)]A^{-\beta}(s)\| \le K(\alpha,\beta,\gamma)\Delta t^{\gamma-\alpha}|t-\tau|^{\beta-\gamma};$$

$$\gamma \in [\beta, \min\{\rho+\varepsilon_1, 1+\alpha\}), \quad \text{if } \rho \ge \tfrac{1}{2};$$

$$\gamma \in [\beta, \min\{\rho+\varepsilon+\delta, 1+\alpha\}), \quad \text{if } \rho < \tfrac{1}{2}.$$

2. Let the operator $A_0 = A(0,v_0)$ be positive definite, self-adjoint, and have a completely continuous inverse. Let $v_0 \in D(A_0^{\beta})$ for some $\beta \in (0,\rho]$. Suppose that for some $\alpha \in [0,\beta)$ and for all $v \in H$ and $t \in [0,T]$ the operator $A(t, A_0^{-\alpha}v)$ is positive definite and self-adjoint, and the domain of definition of the operator $A^{\rho}(t, A_0^{-\alpha}v)$ does not depend on t or v. We shall first assume that $\rho \ge \frac{1}{2}$. Let, for any $v,w \in H$, $\|v\|,\|w\| \le R$,

$$\|A^{\rho}(t+\Delta t, A_0^{-\alpha}v)A^{-\rho}(t, A_0^{-\alpha}w) - A^{-\rho_1}(t+\Delta t, A_0^{-\alpha}v)A^{\rho_1}(t, A_0^{-\alpha}w)\| \le K_2(R)(\Delta t^{\mu} + \|v-w\|^{\nu}), \tag{2}$$

where μ and ν are certain positive numbers ≤ 1.

Consider, in the space C of continuous functions on $[0, t_0]$ $(0 < t_0 \leq T)$ with values in H, the set $Q(K, \eta)$ $(0 < \eta < \beta - \alpha)$ of such functions $v(t)$ that $v(0) = A_0^\alpha v_0$ and

$$\|v(t + \Delta t) - v(t)\| \leq K \Delta t^\eta.$$

The set $Q(K, \eta)$ is closed and convex. By virtue of (2) and Theorem 1, for the operator $A(t, A_0^{-\alpha} v)$ one can construct the operator $U(t, 0; A_0^{-\alpha} v) v_0$. From Theorem 1 and the complete continuity of A_0^{-1} it follows that, for sufficiently small t_0, the operator

$$A_0^\alpha U(t, 0; A_0^{-\alpha} v) v_0$$

maps $Q(K, \eta)$ into its compact part. Finally, this operator is continuous in C, since for any $v, w \in Q(K, \eta)$

$$\|A_0^\alpha [U(t, 0; A_0^{-\alpha} v) - U(t, 0; A_0^{-\alpha} w)] v_0\|_C \leq \overline{K}(\alpha, \beta) \|A_0^\beta v_0\| \, t_0^{\beta - \alpha} \|v - w\|_C.$$

It then follows from Schauder' s fixed-point principle that there exists in $Q(K, \eta)$ a solution of the equation

$$w = A_0^\alpha U(t, 0; A_0^{-\alpha} w) v_0.$$

Hence there follows, for example:

Theorem 2. Let $\mu > 1 - \rho$ and

$$\nu > \frac{1 - \rho}{1 - \alpha}.$$

Then there exists a solution of problem (1), continuous on $[0, t_0]$ and continuously differentiable for $t > 0$. If $\nu = 1$, then this solution may be found by the method of successive approximations.

If, however,

$$\theta = \min\left\{\mu, \frac{(\rho - \alpha)\nu}{1 - \nu}\right\} \leq 1 - \rho,$$

then from Theorem 1 there follows the existence of a function $v(t)$ satisfying, for any $\beta \in (\theta, \rho]$, the equation

$$[A_0^{-\beta} v]' = -A_0^{-\beta} A^\beta(t, v) A^{1-\beta}(t, v) v$$

and the initial condition $v(0) = v_0$.

The case $\rho < \frac{1}{2}$ is considered similarly. For example, if

$$\|A^\rho(t + \Delta t, A_0^{-\alpha} v) A^{-\rho}(t, A_0^{-\alpha} w) - I\| \leq K_3(R)\{\Delta t^\mu + \|v - w\|^\nu\},$$

then Theorem 2 is valid.

Theorem 1 makes it possible, finally, to prove the existence of a solution of an equation more general than (1), $v' + A(t, v)v = f(t, v, Bv)$, if B is an operator subordinated to the operator A_0^α $(0 \leq \alpha < \rho)$.

3. Let G be a domain of m-dimensional space with boundary Γ. Consider the problem*

$$\begin{aligned} v'_t - \left[a_{ik}(t,x,v)v'_{x_k}\right]'_{x_i} + a(t,x,v)v = 0 \qquad (x \in G,\ 0 < t \leq T), \\ v(0,x) = v_0(x), \qquad a_{ik}(t,x,v)v'_{x_k}\cos(n,x_i) + \sigma(t,x,v)v = 0 \qquad (x \in \Gamma). \end{aligned} \tag{3}$$

Assume that the functions $a_{ik}(0,x,v_0)$ are bounded; $a(0,x,v_0) \in L_p(G)$ for some $p > 1$, if $m = 2$, and if $m > 2$, then $a(0,x,v_0) \in L_{m/2}(G)$; $\sigma(0,x,v_0) \in L_p(\Gamma)$ for some $p > 1$, if $m = 2$, and $\sigma(0,x,v_0) \in L_{m-1}(\Gamma)$, if $m > 2$; $a_{ik}(0,x,v_0)\gamma_i\gamma_k \geq a^2\gamma_i\gamma_i$, $a(0,x,v_0) \geq a^2$, $a > 0$, $\sigma(0,x,v_0) \geq 0$. Then in $W_2^1(G)$ there is defined the symmetric bilinear form

$$[v,w] = \int_G a_{ik}(0,x,v_0)v'_{x_k}w'_{x_i}\,dx + \int_G a(0,x,v_0)vw\,dx + \int_\Gamma \sigma(0,x,v_0)vw\,ds.$$

Consider in $L_2(G)$ the operator $\{v - \Delta v\}$, defined on functions $v \in W_2^2(G)$ and satisfying on the boundary the condition $\partial v/\partial n = 0$. Let $C = [I - \Delta]^{1/2}$. Then $D(C) = W_2^1(G)$ and $\|Cv\|^2 = \|v\|^2 + \|\operatorname{grad} v\|^2$.** The expression $[C^{-1}v, C^{-1}w]$ is a linear functional in w and therefore is equal to $(B_0 v, w)$. The operator $A_0 = CB_0C$ is positive definite and self-adjoint, $D(A_0^{1/2}) = W_2^1(G)$, and for any $v \in D(A_0)$ and $w \in W_2^1(G)$, $(A_0 v, w) = [v,w]$. If $a_{ik}(0,x,v_0)v'_{x_k} \in W_2^1(G)$, then hence

$$A_0 v = -\left[a_{ik}(0,x,v_0)v'_{x_k}\right]'_{x_i} + a(0,x,v_0)v$$

in G, and

$$a_{ik}(0,x,v_0)v'_{x_k}\cos(n,x_i) + \sigma(0,x,v_0)v = 0$$

on Γ.

Further,

$$\|A_0^{1/2}v\|^2 \geq a^2(\|\operatorname{grad} v\|^2 + \|v\|^2).$$

This means that A_0^{-1} is completely continuous in $L_2(G)$ and that the operator $A_0^{1/2}$ is strongly invertible, and the latter makes it possible to determine into which spaces from $L_2(G)$ the operators $A_0^{-\alpha}$ $(0 \leq \alpha \leq 1/2)$ act (8). Suppose that for some $\alpha < 1/2$ and any function $y \in L_2(G)$ the function $z = A_0^{-\alpha}y$ has, for all t, the same properties as v_0 at $t = 0$. The set of such functions z contains the entire space $W_2^1(G)$. We shall assume that $v_0 \in W_2^1(G)$. For functions z there is defined an operator $A(t,z)$ possessing the same properties as A_0. Just as in (7), it is shown that for any $v, w \in L_2(G)$, for some $\alpha \in [0, 1/2)$,

$$
\begin{aligned}
&\big\|A^{1/2}(t+\Delta t, A_0^{-\alpha}v)A^{-1/2}(t, A_0^{-\alpha}w) - A^{-1/2}(t+\Delta t, A_0^{-\alpha}v)A^{1/2}(t, A_0^{-\alpha}w)\big\| \\
&\le K_4\Big\{\max_{i,j,k}\big|a_{ik}(t+\Delta t, x, A_0^{-\alpha}v) - a_{ik}(t, x, A_0^{-\alpha}w)\big| \\
&\qquad + \big\|a(t+\Delta t, x, A_0^{-\alpha}v) - a(t, x, A_0^{-\alpha}w)\big\|_{L_{m/2}(G)} \\
&\qquad + \big\|\sigma(t+\Delta t, x, A_0^{-\alpha}v) - \sigma(t, x, A_0^{-\alpha}w)\big\|_{L_{m-1}(\Gamma)}\Big\}.
\end{aligned} \tag{4}
$$

* Here and in what follows summation from 1 to m over repeated indices is understood.

** As C one may take any such self-adjoint operator that $D(C) = W_2^1(G)$ and in $W_2^1(G)$ the norms $\|v\| + \|\operatorname{grad} v\|$ and $\|Cv\|$ are equivalent.

This makes it possible to formulate sufficient conditions for the existence of a solution of problem (1). The solution $v(t,x)$ of this problem we shall call a generalized solution of problem (3). For any function $z \in W_2^1(G)$, the function

$$\int_G v(t,x)z(x)\,dx$$

is continuously differentiable with respect to t, and

$$\frac{d}{dt}\int_G vz\,dx + \int_G a_{ik}(t,x,v)v'_{x_k}z'_{x_i}\,dx + \int_G a(t,x,v)vz\,dx + \int_\Gamma \sigma(t,x,v)vz\,dS = 0.$$

For simplicity we confine ourselves to the formulation of the theorem on the existence of a solution of the problem

$$\frac{\partial v}{\partial t} - \Delta v = 0 \qquad (x \in G,\ 0 < t \le T); \tag{5}$$

$$\frac{\partial v}{\partial n} + \sigma(v)v = 0 \qquad (x \in \Gamma,\ t \in (0,T]), \qquad v(0,x) = v_0(x).$$

Theorem 3. Let $\sigma(v)$ be absolutely continuous. Suppose

$$|\sigma'(v)| \le K_5|v|^r$$

almost everywhere for all $v \in [-b_0, b_0]$, and

$$r < \frac{4-m}{m-2}.$$

Finally, suppose $\sigma'(v) \in L_p[-b_0, b_0]$, $1 < p \le \infty$.

Then there exists a generalized solution of problem (5). If $m = 2, 3$ and $p = \infty$, then this solution can be found by the method of successive approximations.

Proof. Let $\beta = 1 - 1/p$, if $m = 2, 3$, and let

$$\beta < \min\left\{\frac{2}{m-2}, 1 - \frac{1}{p}\right\},$$

if $m > 3$. Then there exists an $\alpha = \alpha(\beta) \in [0, 1/2)$ such that

$$\|\sigma(A_0^{-\alpha} v) - \sigma(A_0^{-\alpha} w)\|_{L_{m-1}(\Gamma)} \leq K(\beta)\|v - w\|^{\beta}_{L_2(G)}.$$

It remains to apply (4) and Theorem 2.

For the case of one spatial variable, stronger results (the existence of a classical solution on the whole segment $[0, T]$) were obtained in ([9]).

We note that the approach presented in this section is also applicable to the first boundary-value problem for a nonlinear parabolic equation.

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