MATHEMATICS

1960

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Abstract

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MATHEMATICS

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ON SOME CLASSES OF SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

(Presented by Academician I. M. Vinogradov on 22 II 1960)

In the book ([1]) various spaces of type S are introduced and considered. We shall examine some spaces of type S from the point of view of the stock of functions in these spaces.

Let the space $S_{l_k}^{m_n}$ consist of infinitely differentiable functions $\varphi(x)$ satisfying the conditions

$$\left|x^k \varphi^{(n)}(x)\right| \leq A^k B^n l_k m_n, \qquad k, n = 0, 1, 2, \dots; \ -\infty < x < \infty, \tag{1}$$

where A and B are constants depending only on the function φ. Below we shall always assume that the sequences $\{l_k\}$ and $\{m_n\}$ are nondecreasing and $l_0 = m_0 = 1$.

Introduce the functions

$$L(x) = \sup_k \frac{|x|^k}{l_k}, \qquad M(x) = \sup_n \frac{|x|^n}{m_n}. \tag{2}$$

Conditions (1) can be written in the equivalent form

$$\left|\varphi^{(n)}(x)\right| \leq B^n \frac{m_n}{L(x/A)}, \qquad n = 0, 1, 2, \dots \tag{3}$$

If the dual space is denoted by $\widetilde{S}_{l_k}^{m_n}$, then, as shown in paper ([2]), $\widetilde{S}_{l_k}^{m_n} \in S_{m_k}^{l_{n+2}}$, and consequently

$$\left|\widetilde{\varphi}^{(n)}(x)\right| \leq A^n \frac{l_{n+2}}{M(x/B_1)}. \tag{4}$$

In paper ([1]) it is shown that, for $|z| \geq 1,\ z = x + iy$, the inequalities

$$|\varphi(z)| \leq \frac{B|z|^2}{A^2 L(|z|/4A)} \int_0^\infty \frac{\operatorname{ch} Byt}{M(t)}\, dt, \tag{5}$$

$$|\widetilde{\varphi}(z)| \leq \frac{A}{M(|z|/B)} \int_0^\infty \frac{\operatorname{ch} Ayt}{L(t)}\, dt. \tag{6}$$

hold.

Simultaneously with the spaces $S_{l_k}^{m_n}$ and $\widetilde{S}_{l_k}^{m_n}$, we shall consider the space S_l^m, consisting of entire functions $f(z)$ subject to the condition

$$|f(z)| \leq \exp\{\, l(A|y|) - m(B|z|)\,\}, \tag{7}$$

where $z = x + iy$; A and B are constants depending on f. We shall assume that the functions $l(x)$ and $m(x)$ satisfy the condition

$$xl'(x) \uparrow \infty, \qquad xm'(x) \uparrow \infty.$$

It is easy to give a necessary and sufficient condition for the nontriviality of the space S_l^m under the assumption that one of the functions l, m satisfies certain regularity requirements on its growth. A condition for the triviality of the space S_l^m is contained in the paper ([3]). We shall give here a result of M. M. Dzhrbashyan in the form of the following lemma.

Lemma 1. *If for every $\theta > 0$*

$$\lim_{x\to\infty} \left[\frac{l(\theta x)}{x} - \int_1^x \frac{m(u)}{u^2}\, du \right] = -\infty,$$

then the space S_l^m is empty.

Let us consider the question of the nontriviality of the space S_l^m. Since, by assumption, $xm'(x) \uparrow \infty$, we may assume that

$$m(x) = \int_0^x \frac{N(u)}{u}\, du,$$

where $N(x)$ is a nondecreasing function taking only integer values. Denote the discontinuity points of the function $N(x)$, arranged in increasing order, by $\rho_1, \rho_2, \dots, \rho_n, \dots$, where each point is repeated as many times as there are units in the corresponding jump.

Lemma 2. *If the function $m(x)$ is such that, for sufficiently large k, as $x \to \infty$*

$$x^{-k} \int_0^x \frac{m(u)}{u^2}\, du \downarrow 0, \tag{8}$$

and if

$$\lim_{x\to\infty}\left[\frac{l(\theta x)}{x}-\int_1^x\frac{m(u)}{u^2}\,du\right]>-\infty,$$

then the space S_l^m *is nontrivial.*

Proof. Let p be an integer. Consider the entire function

$$f(x)=\frac{2}{C_{2p}^p}\sum_{j=1}^p(-1)^{j-1}C_{2p}^{p+j}\frac{\sin jx}{j}.$$

It is not difficult to verify that

$$f'(x)=1-\frac{2^{2p}}{C_{2p}^p}\sin^{2p}\frac{x}{2}.$$

If $x_0,\ 0<x_0<\pi$, is a zero of the function $f'(x)$, then $f(x_0)=\max|f(x)|=M$. Consider the function $g(x)=\frac{1}{M}f(Mx)$. It is clear that $g(x)$ satisfies the conditions

$$g'(0)=1,\qquad |g(x+iy)|\le e^{p|y|}. \tag{9}$$

It is not difficult to verify that in the domain $G_{\eta,\delta}$, defined by the inequalities $|\arg z|\le\delta,\ |\arg z-\pi|\le\delta,\ 0\le|z|\le1+\eta$, the relations

$$\left|\frac{g(z)}{z}\right|\le1,\qquad |g(z)|\le1 \tag{10}$$

hold.

The quantities δ and $\eta>0$ are sufficiently small and depend on p. Choose p so that $2p>k+1$, and consider the function

$$f(z)=\prod_{m=1}^{\infty}\left(\frac{\rho_m g(z/\rho_m)}{z}\right)^2.$$

In view of (8) and the choice of the number p, the infinite product will converge. Let us estimate $|f(z)|$. Suppose that one of the conditions $|\arg z|\leqslant\delta$, $|\arg z-\pi|\leqslant\delta$ is satisfied. Define n from the inequalities $\rho_n\leqslant|z|<\rho_{n+1}$. Then

$$|f(z)|=\left|\frac{\prod_{m=1}^n g^2(z/\rho_m)}{(z^n/\rho_1\rho_2\dots\rho_m)^2}\prod_{n+1}^{\infty}\frac{g^2(z/\rho_m)\rho_m^2}{z^2}\right|<\left|\prod_1^n g^2\left(\frac{z}{\rho_m}\right)\right|e^{-2m(|z|)},$$

since for $m \geqslant n+1$, $z/\rho_m \in G_{\eta,\delta}$. Using (9) and (10), we obtain

$$|f(z)| < \exp\left[2\rho y \sum_{\rho_m \leqslant \frac{|z|}{1+\eta}} \frac{1}{\rho_m} - 2m(|z|)\right].$$

But

$$\sum_{\rho_m \leqslant \frac{|z|}{1+\eta}} \frac{1}{\rho_m} < A_\eta \int_1^y \frac{m(t)}{t^2}\, dt + \frac{A_\eta}{y}\, m\left(\frac{|z|}{1+\eta/2}\right).$$

Therefore

$$|f(z)| \leqslant \exp\left\{2\rho A_\eta y \int_1^y \frac{m(u)}{u^2}\, du - m(|z|) + C\right\}. \tag{11}$$

Let us now consider the case when $\delta < \arg z < \pi - \delta$ or $\pi + \delta < \arg z < 2\pi - \delta$. Since $\log \dfrac{g(z)}{z} = O(z^{2p})$ for $|z| \leqslant 1$, it follows that

$$|f(z)| \leqslant \exp\left\{2\rho y \sum_1^n \frac{1}{\rho_m} + C_1 |z|^{2p} \sum_{n+1}^{\infty} \frac{1}{\rho_m^{2p}} - 2m(|z|)\right\}.$$

But

$$\sum_{n+1}^{\infty} \frac{1}{\rho_m^{2p}} < (2p+1)^3 \int_{|z|}^{\infty} \frac{dt}{t^{2p}} \int_1^t \frac{m(u)}{u^2}\, du.$$

Applying (8), we obtain

$$|f(z)| \leqslant \exp\left\{C_2(z) \int_1^{2|z|} \frac{m(u)}{u^2}\, du - m(|z|)\right\}. \tag{12}$$

From inequalities (11) and (12) it follows that, for any z, the inequality

$$|f(z)| < \exp\left\{\sqrt{\theta}\, |y| \int_1^{\sqrt{\theta}|y|} \frac{m(u)}{u^2}\, du - m(|z|) + C\right\}$$

holds, where θ is a constant depending on p. By the condition of the lemma,

$$|f(z)| < \exp\{l(\theta|y|) - m(|z|) + D\},$$

where D is a constant. Thus the space S_l^m is nontrivial.

Theorem 1. *Let $m(x)$ satisfy condition (8). In order that the space S_l^m be nontrivial, it is necessary and sufficient that, for at least one $\theta > 0$,*

$$\lim_{x\to\infty}\left[\frac{l(\theta x)}{x} - \int_1^x \frac{m(u)}{u^2}\,du\right] > -\infty.$$

Let the function $L(x)$ be such that $\frac{d}{dx}\log L(x) \uparrow \infty$. Then, if we denote by $\psi(x)$ the function inverse to $\frac{d}{dx}\log L(x)$, and set

$$l(x) = \int_0^x \psi(t)\,dt, \tag{13}$$

we immediately obtain that the question of nontriviality of the space $S_{l_k}^{m_r}$ is equivalent to the question of nontriviality of the space S_l^m, where $l(x)$ is defined by (13), and $m(x) = \log M(x)$.

Consider the case when $M(x)$ is a function of infinite order, i.e.,

$$\overline{\lim_{x\to\infty}}\frac{\log\log M(x)}{\log x} = \infty. \tag{14}$$

If (14) holds, we shall require that

$$\underline{\lim}_{x\to\infty}\frac{\log\log M(x)}{\log x} > 1. \tag{15}$$

Then the function

$$\int_1^\infty \frac{\operatorname{ch} xt}{M(t)}\,dt$$

is of finite order, and for the nontriviality of the class it is necessary that $L(x)$ also be of finite order. We shall say that the functions $L(x)$ and $M(x)$ satisfy condition (α) if the following relations are fulfilled:

1. If $M(x)$ is of finite order, then there always exists a k such that

$$x^{-k}\int_0^x \frac{\log M(u)}{u^2}\,du \downarrow 0, \tag{16}$$

   and the function $L(x)$ satisfies the condition

$$\frac{d}{dx}\log L(x) \uparrow \infty.$$

2. If $M(x)$ satisfies (14), then (15) also holds, and item 1 holds for the functions $L(x)$ and $M(x)$.

From Theorem 1 it follows

Theorem 2. *If $L(x)$ and $M(x)$ satisfy condition (α), then the space $S_{l_k}^{m_n}$ is nontrivial if and only if, for at least one $\theta > 0$,*

$$\underline{\lim} \left[\frac{l(\theta x)}{x} - \int_0^x \frac{\log M(u)}{u^2} \, du \right] > -\infty,$$

where $l(x)$ is defined by formula (13).

Finally, the conditions for nontriviality of the space $S_{l_k}^{m_n}$ can be formulated in the following interesting way. Let $\mu(x)$ be the function inverse to the function

$$\int_0^x \frac{\log M(u)}{u^2} \, du.$$

Theorem 3. *If for every $\theta > 0$*

$$\lim_{x\to\infty} \frac{L(\theta x)}{M[|\mu(x)|]} = \infty, \tag{17}$$

then the space $S_{l_k}^{m_n}$ is trivial. If the functions $L(x)$, $M(x)$ satisfy condition (α) and for at least one $\theta > 0$

$$\lim_{x\to\infty} \frac{L(\theta x)}{M(\mu(x))} < \infty, \tag{18}$$

then $S_{l_k}^{m_n}$ is nontrivial.

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Received
11 II 1960

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Note: Figure translations are in progress. See original paper for figures.