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Abstract

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ON THE REGULARITY OF SOLUTIONS OF MULTIDIMENSIONAL ELLIPTIC EQUATIONS AND VARIATIONAL PROBLEMS

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1. We consider the problem of minimizing the functional

$$I(u) = \int_\Omega F(x, u, u_x)\, dx \quad \text{under the condition} \quad u|_S = \varphi(s), \tag{1}$$

where $x = (x_1, \dots, x_n)$; Ω is a bounded domain of n-dimensional space with smooth boundary; $\varphi(s)$ is a prescribed function on the boundary S; $n \geq 2$. Throughout the paper we shall assume that F is twice continuously differentiable with respect to its arguments, and that its second derivatives satisfy a Hölder condition with exponent $\alpha > 0$ in the variables x, u, u_x for $x \in \Omega$ and for arbitrary bounded values of u and u_x.

Let, moreover, F satisfy the regularity condition

$$a_{ij}\xi_i\xi_j \equiv F_{u_{x_i}u_{x_j}}\xi_i\xi_j \geq m \sum_{i=1}^{n} \xi_i^2, \quad m > 0. \tag{2}$$

If $\varphi(s)$ is extended to Ω in such a way that $\varphi(x) \in W_p^1(\Omega)$, $p > 1$, and F satisfies the additional requirements

$$F \leq M_1(|\operatorname{grad} u|^p + 1); \tag{3}$$

$$F \geq m_1|\operatorname{grad} u|^p + L, \quad m_1 > 0, \tag{4}$$

then, as Morrey showed ([1]), there exists a generalized solution of the problem in the class of functions satisfying the condition $u - \varphi \in \overset{\circ}{W}{}_p^1(\Omega)$. The differential properties of solutions of the variational problem (1) for $n \geq 2$ were studied in works ([2–6]). In ([2]) it is shown that if the first derivatives of the generalized solution are continuous, then $u \in C_{2,\alpha}(\Omega)$. For $n = 2$ Morrey ([1]) obtained a stronger assertion: if a generalized solution of the variational problem satisfies a Lipschitz condition, then $u \in C_{2,\alpha}(\Omega')$, $\Omega' \subset \Omega$. G. I. Shilova ([4]) proved the continuity of a generalized solution if the order p of growth of F with respect to the first derivatives is equal to n ($n \geq 2$) (for $p > n$ this result follows from the embedding theorems of S. L. Sobolev). O. A. Ladyzhenskaya ([5]) showed that if F satisfies conditions (2)–(4), and also

$$F_{u_{x_i} u_{x_j}} \xi_i \xi_j \geq m_2(|\operatorname{grad} u|^{p-2} + 1) \sum_{i=1}^{n} \xi_i^2, \quad m_2 > 0; \tag{5}$$

$$\left| \frac{\partial^\beta F}{\partial u_{x_1}^{k_1} \cdots \partial u_{x_n}^{k_n} \partial u^l \partial x_1^{i_1} \cdots \partial x_n^{i_n}} \right| \leq M_2(|\operatorname{grad} u|^{p-l-\sum_{s=1}^{n} k_s} + 1), \tag{6}$$

$$\beta = 0, 1, 2; \quad l = 0, 1; \quad p \geq 2,$$

then the generalized solution of the variational problem has generalized derivatives of the second order and almost everywhere satisfies the Euler equation; if, moreover, F does not depend on u and x for large values of $|\operatorname{grad} u|$, then u is continuous and has bounded first derivatives. For the case where F depends only on the first derivatives and $p = 2$, De Giorgi ([6]) proved that the generalized solution will belong to $C_{2,\alpha}(\Omega')$, $\Omega' \subset \Omega$.

In the present paper the following theorem is proved, which is a generalization of the indicated theorem of Morrey to the case of arbitrary n.

Theorem 1. *If the generalized solution of the variational problem (1) has bounded first derivatives (or, what is the same, satisfies the Lipschitz condition) and F satisfies the smoothness conditions indicated above and the regularity condition (2), then $u \in C_{2,\alpha}(\Omega')$, where Ω' is any interior subdomain of Ω. If $\varphi \in C_{2,\alpha}(S)$ and the boundary S has second derivatives satisfying the Hölder condition with exponent α, then $u \in C_{2,\alpha}(\Omega)$.*

The proof is based on two lemmas, which are generalizations of De Giorgi' s lemmas ([6]).

Let us introduce the following notation: $I(\rho)$ is a ball of radius ρ; $\operatorname{osc}(v; \rho)$ is the oscillation of the function v in $I(\rho)$; $A_{k,\rho}$ and $B_{k,\rho}$ are the intersections of $I(\rho)$ with the sets where $v > k$ and $v < k$, respectively.

We shall say that v belongs to the class $\mathfrak{B}(\Omega, \gamma, M)$ if $v \in W_2^1(\Omega)$, $|v| \leq M$, and for any ρ, k, and σ, for which $I(\rho) \subset \Omega$, $|k| \leq M$, $0 < \sigma \leq 1$, the inequalities hold

$$\int_{A_{k,\rho-\sigma\rho}} (\operatorname{grad} v)^2 \, dx \leq \gamma \left[\frac{1}{\sigma^2 \rho^2} \int_{A_{k,\rho}} (v-k)^2 \, dx + \operatorname{mes} A_{k,\rho} \right], \tag{7}$$

$$\int_{B_{k,\rho-\sigma\rho}} (\operatorname{grad} v)^2\,dx \le \gamma \left[\frac{1}{\sigma^2\rho^2} \int_{B_{k,\rho}} (v-k)^2\,dx + \operatorname{mes} B_{k,\rho} \right]. \tag{7'}$$

Here $\gamma = \text{const} > 0$.

Lemma 1. *For any $v \in \mathfrak{B}(\Omega,\gamma,M)$, ρ, σ $(0 < \sigma < 1)$, there exists $\theta(\sigma)$ such that, if $I(\rho) \subset \Omega$, then from $\operatorname{mes} A_{k,\rho} < \theta(\sigma)\rho^n$ it follows that $\operatorname{mes} A_{k+\sigma H+\sigma^2\rho,\rho-\sigma\rho} = 0$, where $H = \max_{x\in A_{k,\rho}}(v-k)$.*

Lemma 2. *For any $v \in \mathfrak{B}(\Omega,\gamma,M)$ there exist constants C,α $(0 < \alpha \le 1)$ such that, if $I(\rho) \subset \Omega$, then $\operatorname{osc}(u;\rho) \le C\rho_0^{-\alpha}\rho^\alpha$, where ρ_0 is the distance from the center of the sphere $I(\rho)$ to the boundary of the domain Ω.*

We shall show that the first derivatives of the generalized solution u of problem (1) belong to the class $\mathfrak{B}(\Omega,\gamma,M)$, if they are bounded in Ω. The function u satisfies the integral identity

$$\int_\Omega (F_{u_{x_i}}\eta_{x_i} + F_u\eta)\,dx = 0 \tag{8}$$

for any $\eta \in \overset{0}{W}{}_2^1(\Omega)$. Hence, using the device of O. A. Ladyzhenskaya ([5]), we obtain that

$$\int_\Omega \sum_{i,l=1}^n u_{x_ix_l}^2\,dx \le \text{const}$$

and for any $\eta \in \overset{0}{W}{}_2^1(\Omega)$ the identity holds

$$\int_\Omega (a_{ij}u_{x_jx}{}_l + F_{u_{x_i}u}u_{x_l} + F_{u_{x_i}x_l})\eta_{x_i}\,dx = \int_\Omega F_u\eta_{x_l}\,dx, \qquad l = 1,\dots,n. \tag{9}$$

Let $I(\rho) \subset \Omega$,

$$\eta(x) = \begin{cases} [u_{x_l}(x) - k]\xi(\rho') & \text{if } u_{x_l} \ge k, \\ 0 & \text{if } u_{x_l} \le k, \end{cases}$$

where ρ' is the distance from x to the center of the sphere $I(\rho)$,

$$\zeta(\rho') = \begin{cases} 1, & \text{if } \rho' \leqslant \rho - \sigma\rho; \\ (\rho-\rho')^2/\sigma^2\rho^2, & \text{if } \rho - \sigma\rho \leqslant \rho' \leqslant \rho; \\ 0, & \text{if } \rho' \geqslant \rho. \end{cases}$$

It is obvious that $\eta(x) = 0$, if $x \bar{\in} A_{k,\rho}$.
From (9) it follows that

$$\int_{A_{k,\rho}} \left[a_{ij} u_{x_j x^i} u_{x_l x^i} \zeta + a_{ij} u_{x_j} (u_{x_l} - k) \zeta_{x_i}\right] dx =$$

$$= \int_{A_{k,\rho}} \text{bounded function} \cdot \left[u_{x_l x^i} \zeta + (u_{x_l} - k) \zeta_{x_i}\right] dx. \tag{10}$$

Hence, using (2) and the inequality $ab \leqslant a^2/2\varepsilon + \varepsilon b^2/2,\ \varepsilon > 0$, we obtain

$$\int_{A_{k,\rho-\sigma\rho}} \sum_{i=1}^{n} u^2_{x_l x^i} \, dx \leqslant \gamma \left[\frac{1}{\sigma^2 \rho^2} \int_{A_{k,\rho}} (u_{x_l} - k)^2 \, dx + \operatorname{mes} A_{k,\rho}\right]. \tag{11}$$

Similarly,

$$\int_{B_{k,\rho-\sigma\rho}} \sum_{i=1}^{n} u^2_{x_l x^i} \, dx \leqslant \gamma \left[\frac{1}{\sigma^2 \rho^2} \int_{B_{k,\rho}} (u_{x_l} - k)^2 \, dx + \operatorname{mes} B_{k,\rho}\right]. \tag{11'}$$

The investigation of the solution in a closed domain required additional considerations.

2. Lemmas 1, 2, Theorem 1, and the estimates of the first derivatives from ([5], [7]) made it possible to investigate the question of when the variational problem has a classical solution. With the aid of these same lemmas and of the estimates of first derivatives from ([7]), the existence of a classical solution is established for the first boundary-value problem for quasilinear elliptic equations of the form

$$\frac{\partial}{\partial x_i}(a_i(x, u, u_x)) + a(x, u, u_x) = 0. \tag{12}$$

The Euler equation for the variational problem (1) is a special case of equations of this kind. All restrictions on a_i and a are due, in the main, only to the desire to obtain estimates of $\max|u|$ and $|u_x|$. Known are cases when $\max|u|$ is estimated; we shall not present them here. The estimate of the first derivatives of solutions is carried out by the method given in ([7]), under the following conditions:

$$m_3(|\operatorname{grad} u|^{p-2}+1) \sum_{i=1}^{n} \xi_i^2 \leqslant \frac{\partial a_i}{\partial u_{x_j}} \xi_i \xi_j \leqslant M_3(|\operatorname{grad} u|^{p-2}+1) \sum_{i=1}^{n} \xi_i^2, \qquad m_3 > 0;$$

$$\left|\frac{\partial^2 a_i}{\partial u_{x_j} \partial u_{x_k}}\right| \leqslant M_4(|\operatorname{grad} u|^{p-3} + 1), \qquad p \geqslant 2;$$

$$\left|\frac{\partial^2 a_i}{\partial u_{x_j} \partial x_k}\right| \leqslant M_5(|\operatorname{grad} u|^{p-2}+1); \tag{I}$$

$$\left|\frac{\partial^2 a_i}{\partial u\, \partial x_k}\right|, \quad \left|\frac{\partial a_i}{\partial u}\right|, \quad \left|\frac{\partial a}{\partial u_{x_k}}\right| \leqslant M_6(|\operatorname{grad} u|^{p-1}+1);$$

$$\left|\frac{\partial^2 a_i}{\partial x_k \partial x_j}\right|, \quad \left|\frac{\partial a_i}{\partial x_k}\right|, \quad \left|\frac{\partial a}{\partial x_k}\right|, \quad |a| \leqslant M_7(|\operatorname{grad} u|^{p}+1).$$

These conditions are natural. They are satisfied, for example, in those cases when a_i and a behave like polynomials with respect to the arguments u_{x_k}. In addition, one of the following requirements must be satisfied:

a) there exists $R \geqslant 0$ such that, if $|\operatorname{grad} u| > R$, then

$$\left|\frac{\partial^2 a_i}{\partial u_{x_j} \partial u}\right| \ll \delta |\operatorname{grad} u|^{p-2};$$

$$\left|\frac{\partial^2 a_i}{\partial u^2}\right| \ll \delta |\operatorname{grad} u|^{p-1}; \tag{II}$$

$$\left|\frac{\partial a}{\partial u}\right| \ll \delta |\operatorname{grad} u|^{p} \quad \text{or} \quad \frac{\partial a}{\partial u} \ll 0,$$

where δ is less than a certain number.

b) $\operatorname{osc}(u; \Omega) \ll 1$.

If conditions (I) and (II) (a) or b)) are fulfilled, we guarantee the existence of a classical solution of the first boundary-value problem, provided only that $\max |u|$ can be estimated.

3. Finally, we have considered a quasilinear elliptic equation of the form

$$a_{ij}(x,u) u_{x_i x_j} = f(x,u,u_x) \quad \text{with} \quad u|_S = \varphi(s). \tag{13}$$

In the work [8] the solvability of problem (13) was proved under certain conditions, caused by the fact that the author uses estimates of $\max |u_x|$ from the work [7]. Let us note that, as far as we know, in all works on multidimensional quasilinear equations of elliptic type [7,9] there are some restrictions on the occurrence of u itself in the coefficients of the equation. They have not been removed even for $n = 2$ [10,11]. We remove these restrictions and prove the solvability of problem (13) in all those cases when it is possible to estimate $\max |u|$ and when a_{ij} and

f, for $x \in \Omega$, have bounded derivatives with respect to their arguments in any bounded domain of the space (u, u_x), while f satisfies the conditions

$$\max\left\{|f|, \left|\frac{\partial f}{\partial u}\right|, \left|\frac{\partial f}{\partial x}\right|\right\} \ll C_1\left(|\operatorname{grad} u|^2 + 1\right);$$

$$\left|\frac{\partial f}{\partial u_{x_k}}\right| \ll C_2\left(|\operatorname{grad} u| + 1\right).$$

If f is linear with respect to u_{x_k}, then it is sufficient to require that f satisfy a Hölder condition in x, u.

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Note: Figure translations are in progress. See original paper for figures.