L. P. Nizhnik

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Abstract

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L. P. Nizhnik

THE SCATTERING PROBLEM UNDER A NONSTATIONARY PERTURBATION

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Mathematics

The problem of scattering of plane waves for the equation

$$\Box u(x,t) + c(x,t)u(x,t) = 0, \tag{1}$$

where x is a point of three-dimensional Euclidean space E^3; $\Box = \Delta - \partial^2/\partial t^2$, may be posed as follows: it is required to find a solution of equation (1) of the form

$$u = u(x,t;\omega,\mu) = e^{i\omega(\mu\cdot x - t)} + v(x,t;\omega,\mu),$$

where μ is the unit vector of the direction of the plane wave $e^{i\omega(\mu\cdot x - t)}$, and v, as $|x| \to \infty$, satisfies the Fock radiation conditions ([1], p. 435; [2])

$$v = O\left(\frac{1}{|x|}\right), \qquad \frac{\partial v}{\partial t} = O\left(\frac{1}{|x|}\right), \qquad |\operatorname{grad} v| = O\left(\frac{1}{|x|}\right), \qquad \frac{\partial v}{\partial |x|} + \frac{\partial v}{\partial t} = O\left(\frac{1}{|x|}\right). \tag{2}$$

Here we shall assume that conditions (2) are fulfilled uniformly for $t \in (-\infty, \infty)^*$.

If $c(x,t)$ does not depend on t, then equation (1), by separation of variables, reduces to the stationary Schrödinger equation $-\Delta u + cu = k^2 u$, and the Fock radiation conditions in this case become the well-known Sommerfeld radiation conditions. The scattering problem for the Schrödinger equation was solved in ([3]); the inverse problem was studied in ([4–6]).

In the present note it will be shown that, under certain conditions of smoothness and decay at infinity imposed on the function $c(x,t)$, there exists a unique solution of the scattering problem for equation (1). In addition, the question of reconstructing the function $c(x,t)$ from scattering data is considered.

The results presented can be carried over to the equation $[\Box - m^2 + c(x,t)]u = 0$. In this case the extraction of outgoing waves by means of the Fock radiation conditions must be replaced by the limiting absorption principle ([7], p. 499). In this case the scattering of plane waves can be interpreted as the scattering of mesons by a nonstationary force center.

1°. Everywhere below we shall assume that $c(x,t)$, together with all partial derivatives up to and including the second order, are continuous functions which are majorized by the function

$$\frac{K(t)}{1+|x|^{3+\varepsilon}},$$

where $\varepsilon > 0$, and the function $K(t)$ is uniformly bounded for $t \in (-\infty, \infty)$.

By $C(\Omega)$ we shall denote the Banach space of continuous functions on Ω with the uniform norm. $\Box_r^{-1}$ will denote the inverse of the operator $\Box$ by means of the retarded Green function:

$$\Box_r^{-1} f(x,t) = -\frac{1}{4\pi}\int \frac{f(s, t-|x-s|)}{|x-s|}\, ds.$$

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* One could have assumed that conditions (2) are fulfilled for $t = t_0 - |x|$ uniformly in t_0 from an arbitrary fin

If there exists a solution of the scattering problem $u = e^{i\omega(\mu\cdot x - t)} + v$ for equation (1), then $\Box v + cv = -ce^{i\omega(\mu\cdot x-t)}$. Since v satisfies the radiation conditions, it follows that

$$v = -\Box_r^{-1} c e^{i\omega(\mu\cdot x - t)} - \Box_r^{-1} cv. \tag{3}$$

Conversely, every sufficiently smooth solution of equation (3), together with the plane wave $e^{i\omega(\mu\cdot x-t)}$, gives a solution of the scattering problem. For brevity, denote the operator

$$-\Box_r^{-1} cv = Av.$$

Lemma 1. *If there exists a unique solution of the equation $w = h + Aw$ in $C(E^4)$ for every right-hand side h, then there exists a unique solution of the scattering problem.*

Proof. The uniqueness of the solution of the scattering problem follows from the uniqueness of the solution of the equation $w = Aw$ in $C(E^4)$. Suppose there exists a solution of the equation $w = h + Aw$ in $C(E^4)$ for every right-hand side h. This means that the operator $(I - A^*)^{-1}$ is bounded. Consider the operator $D_m = -d^2/dt^2 + m^2$. This operator has in $C(E^4)$ a bounded inverse D_m^{-1}. The operator

$$D_{mAD}m^{-1} = A - 2\Box_r^{-1} c_t \frac{\partial}{\partial t} D_m^{-1} - \Box_r^{-1} c_{tt} D_m^{-1} = A + B_m,$$

where B_m is a bounded operator which, for large m, can be made arbitrarily small in norm. Therefore, for large m there exists a bounded operator

$$(I - A^* - B_m^*)^{-1} = [I - (D_{mAD} m^{-1})^*]^{-1},$$

i.e., the equation $\varphi = f + D_{mAD} m^{-1}\varphi$ has a solution in $C(E^4)$ for every right-hand side f. If we put $D_m^{-1}\varphi = w$, then $w = D_m^{-1} f + Aw$. Thus, for sufficiently smooth free terms h there exist smooth solutions of the equation $w = h + Aw$. If we choose $h = Ae^{i\omega(\mu \cdot x - t)}$, then we obtain that the solution of equation (3) exists and is a sufficiently smooth function, i.e., there exists a solution of the scattering problem.

Theorem 1. *For small $c(x,t)$ $(K(t) < \varepsilon/3)$ there exists a unique solution of the scattering problem for equation (1).*

The proof follows from Lemma 1 and from the fact that, for small $c(x,t)$, $\|A\| < 1$.

2°. In this subsection we shall assume that

$$K(t) \leq \frac{C}{1 + |t|^{1+\delta}}, \qquad \delta > 0.$$

By $L_2(p(x))$ we shall denote the Hilbert space with norm

$$\|f(x,t)\|^2 = \int |f(x,t)|^2 p(x)\, dx\, dt.$$

Lemma 2. *If, for some $a > 0$, there exists a unique solution of the equation $w = h + Aw$ in $L_2((1 + |x|)^{-1-a})$ for every right-hand side h, then there exists a unique solution of the scattering problem for equation (1).*

The proof of this lemma is completely analogous to the proof of Lemma 1.

Lemma 3. *For $\varepsilon > 1/2 + \frac{1}{2}a$, the operator A is completely continuous in $L_2((1 + |x|)^{-1-a})$.*

Indeed, consider the operator

$$A_\xi f(x,t) = \frac{1}{4\pi} \int \frac{e^{-\xi|x-s|}}{|x-s|} c(s, t - |x-s|) f(s, t - |x-s|)\, ds$$

for $\xi \geq 0$, $A_0 = A$. It is easy to prove that the operator A_ξ for $\xi \geq 0$ is bounded in $L_2((1 + |x|)^{-1-a})$, and

$$\|A_\xi - A_0\| \to 0$$

as $\xi \to 0$. The operator A_ξ for $\xi > 0$ is completely continuous in $L_2((1+|x|)^{-1-a})$. For this it suffices to show that the operator

$$A'_\xi f = \frac{1}{4\pi} \int \frac{e^{-\xi|x-s|}}{|x-s|} c(s, t - |x-s|)(1 + |s|)^{1/2+1/2a} f(s, t - |x-s|)\, ds$$

is completely continuous in $L_2(E^4)$. But the unitarily equivalent operator $T_\xi = UA'_\xi U^{-1}$, where U is the Fourier transform, has the form $(T\varphi)(p) =$

$$= (T_\xi\varphi)(p, p_0) = \frac{1}{p^2 - (p_0 - i\xi)^2} \int b(p-q)\varphi(q)\, d_4q$$

($b(p)$ is the Fourier transform of $c(x,t)$) $(1+|x|)^{1/2+1/2a}$, and the complete continuity of the operator T_ξ is verified analogously to Lemma 2 of [8]. Consequently, A'_ξ is completely continuous, and hence so is the operator A_ξ for $\xi > 0$. Since $\|A_\xi - A_0\| \to 0$ as $\xi \to 0$, the operator $A_0 = A$ is also completely continuous.

Lemma 4. *If $\varepsilon > 1/2 + 1/2a$, then the equation $w = Aw$ in $L_2((1+|x|)^{-1-a})$ has only the trivial solution $w = 0$.*

Indeed, suppose there is a nontrivial solution of the equation $w = Aw$. Denote it by w_0. Consider the operator A on functions from the half-space $t \le T$. For sufficiently negative T, $\|A\| < 1$, and therefore $w_0 = 0$ for such t. Let T_0 be the largest number for which $w_0 = 0$ for $t \le T_0$. Estimating the equality $w_0 = Aw_0$, we obtain

$$\int_{t \le T_0+\Delta} |w_0(x,t)|^2 (1+|x|)^{-1-a}\, dx\, dt \le$$
$$\le C_1\Delta^2 \int_{t \le T_0+\Delta} |w_0(x,t)|^2 (1+|x|)^{-1-a}\, dx\, dt.$$

If Δ is a sufficiently small number, then $w_0 = 0$ for $t \le T_0 + \Delta$. This contradicts the choice of T_0.

From the general theory of equations in Banach spaces and Lemmas 2, 3, and 4 it follows:

Theorem 2. *For $\varepsilon > 1/2$ there exists a unique solution of the scattering problem for equation (1).*

3°. It is easy to prove that the solution of the scattering problem has the following asymptotic form as $|x| \to \infty$:

$$u(x,t;\omega,\mu) = e^{i\omega(\mu\cdot x - t)} + \frac{\rho(t-|x|,\nu;\omega,\mu)}{|x|} + o\left(\frac{1}{|x|}\right),$$

where ν is the unit vector in the direction of the vector x; $\rho(\tau,\nu;\omega,\mu)$ is a uniformly bounded function of its arguments.

Passing to the Fourier transform of the function ρ with respect to the first argument, we obtain

$$u(x,t;\omega,\mu) = e^{i\omega(\mu\cdot x - t)} + \int \frac{e^{i\gamma(|x|-t)}}{|x|} F(\gamma,\nu;\omega,\mu)\, d\gamma + o\left(\frac{1}{|x|}\right),$$

where $F(\gamma, \nu; \omega, \mu)$, as the Fourier transform of a bounded function, is a generalized function of measure type, and therefore

$$\int \frac{e^{i\gamma(|x|-t)}}{|x|} F(\gamma, \nu; \omega, \mu)\, d\gamma$$

represents a sum of outgoing spherical waves of various frequencies. If the function $c(x, t)$ satisfies the conditions of item **2°**, then this sum will be continuous. In the stationary case, when $F(\gamma, \nu; \omega, \mu) = \delta(\gamma - \omega) f(\nu, \omega, \mu)$, the integral under consideration reduces to a single term

$$\frac{e^{i\omega(|x|-t)}}{|x|} f(\nu, \omega, \mu),$$

where $f(\nu, \omega, \mu)$ is the usual scattering amplitude. The function $F(\gamma, \nu; \omega, \mu)$, by analogy with the stationary case, will be called the scattering amplitude.

Using the method of [4,6], let us consider the question of reconstructing the function $c(x, t)$ from the scattering amplitude.

Theorem 3. *From the scattering amplitude $F(\gamma, \nu; \omega, \mu)$ one uniquely reconstructs the Fourier transform of the function $c(x, t)$ outside the light cone $p^2 - p_0^2 \geq 0$. Namely ($\bar{p}^2 - p_0^2 \geq 0$),*

$$\tilde{c}(\bar{p}, p_0) = \frac{1}{4\pi^2} \int c(x, t) e^{i(\bar{p}\cdot x + p_0 t)}\, dx\, dt = \frac{1}{\pi} \lim_{\substack{\omega \to \infty \\ \omega\mu - (\omega + p_0)\nu = \bar{p}}} F(\omega + p_0, \nu; \omega, \mu).$$

Proof. On the basis of Lemma 1 [9] (or Lemma 1.3 [5]) one can show that, for large ω, $v = Ae^{i\omega(\mu\cdot x - t)} + O\left(\frac{1}{\omega}\right)$. Consequently,

$$\begin{aligned} F(\gamma, \nu; \omega, \mu) &= \frac{1}{4\pi} \int e^{-i\gamma(\nu\cdot x - t)} C(x, t) e^{i\omega(\mu\cdot x - t)}\, dx\, dt + O\left(\frac{1}{\omega}\right) \\ &= \frac{1}{4\pi} \int C(x, t) e^{i(\bar{p}\cdot x + p_0 t)}\, dx\, dt + O\left(\frac{1}{\omega}\right), \end{aligned}$$

where $p_0 = \gamma - \omega$, $\bar{p} = \omega\mu - \gamma\nu$. Passing to the limit as $\omega \to \infty$, we obtain the required reconstruction formula.

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Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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References

1. V. A. Fock, *The Theory of Space, Time, and Gravitation*, 1955.

2. I. T. Todorov, *Uspekhi Mat. Nauk*, **13**, issue 2(80), (1958).

3. A. Ya. Povzner, *Mat. Sbornik*, **32** (74), No. 1 (1953).

4. E. E. Levin, Abstract of a Candidate' s dissertation, Moscow State University, 1955.

5. Yu. M. Berezanskii, *Trudy Moskov. Mat. Obshch.*, **7**, 3 (1958).

6. L. D. Faddeev, *Vestnik Leningrad. Univ.*, No. 7 (1956).

7. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, 1951.

8. L. P. Nizhnik, *Dokl. Akad. Nauk SSSR*, **124**, No. 3 (1959).

9. L. D. Faddeev, *Vestnik Leningrad. Univ.*, No. 7 (1957).

Note: Figure translations are in progress. See original paper for figures.