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Abstract

Full Text

Mathematics

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On an Approximate Method for Constructing the Cauchy Function

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Consider the equation

$$L[y] \equiv y^{[n]} - \sum_{k=0}^{n-1} g_k(x) y^{(k)} = f(x), \qquad y^{(k)}(a) = y_0^{(k)}, \quad k = 0, \dots, n-1, \quad (1)$$

where g_k and f are continuous on $[a, b)$. Let $K(x, s)$ be the Cauchy function of the operation $L[y]$, i.e.

$$K(x,s) = \begin{vmatrix} u_0(s) & \cdots & u_{n-1}(s) \\ \cdot & \cdots & \cdot \\ u_0^{(n-2)}(s) & \cdots & u_{n-1}^{(n-2)}(s) \\ u_0(x) & \cdots & u_{n-1}(x) \end{vmatrix} : \begin{vmatrix} u_0(s) & \cdots & u_{n-1}(s) \\ \cdot & \cdots & \cdot \\ u_0^{(n-2)}(s) & \cdots & u_{n-1}^{(n-2)}(s) \\ u_0^{(n-1)}(s) & \cdots & u_{n-1}^{(n-1)}(s) \end{vmatrix},$$

where $u_k(x)$ $(k = 0, \dots, n-1)$ is a fundamental system of solutions of the equation $L[y] = 0$.

The problem of constructing $K(x, s)$ is equivalent to the problem of constructing the fundamental system u_k. The solution u of equation (1) can be represented in the form

$$u(x) = v(x) + \int_a^x K(x,s) f(s)\, ds, \quad (2)$$

where $v(x)$ is the solution of equation (1) for $f(x) \equiv 0$. Thus, constructing the Cauchy function solves the problem of integrating equation (1). Below we propose a process of successive approximations to the function $K(x, s)$, distinguished by an extraordinarily rapid rate of convergence.

Let $W(x, s)$ be a function, continuously differentiable n times with respect to x for $a \le s \le x < b$, satisfying the condition: $W^{(k)}(s, s) = \delta_{k,n-1}$ $(k = 0, \dots, n-1$; $\delta_{i,j}$ is the Kronecker symbol). Define the sequence $\{W_i(x, s)\}$ as follows:

$$W_0(x,s) = W(x,s),$$

$$W_{i+1}(x,s) = W_i(x,s) - \int_s^x W_i(x,t) L[W_i(t,s)]\,dt \qquad (i \geq 1).$$

When considering functions $\psi(x,s)$ depending on the parameter s, we denote by $\psi^{(k)}(x,s)$ the corresponding derivative with respect to the first argument. In this case

$$L[\psi(x,s)] \equiv \psi^{(n)}(x,s) - \sum_{k=0}^{n-1} g_k(x)\psi^{(k)}(x,s).$$

The rate of convergence of the sequence $\{W_i\}$ is determined by

Theorem 1. *Let*

$$\|L[W_0(x,s)]\| \leq (x-s)^\beta Q,$$

$$|K^{(k)}(x,s) - W_0^{(k)}(x,s)| \leq (x-s)^{\alpha_k} P_k \qquad (k = 0,\dots,n).$$

Then

$$|K^{(k)}(x,s) - W_i^{(k)}(x,s)| \leq \frac{P_k \alpha_k! (Q\beta!)^{2^i-1}}{[(2^i-1)(\beta+1)+\alpha_k]!} (x-s)^{(2^i-1)(\beta+1)+\alpha_k}.$$

Let $z(x)$ $(z^{(k)}(a) = y_0^{(k)},\ k = 0,\dots,n-1)$ be some function n times continuously differentiable on $[a,b]$. Then for the approximate solution

$$\tilde{u}(x) = z(x) + \int_a^x W_i(x,s)\{f(s) - L[z(s)]\}\,ds \tag{3}$$

of equation (1), we have, by virtue of the theorem just stated, the estimate: if $|f(x) - L[z(x)]| \leq (x-s)^\gamma R$, then

$$|\tilde{u}^{(k)}(x) - u^{(k)}(x)| \leq \frac{R\gamma! P_k \alpha_k! (Q\beta!)^{2^i-1}}{[(2^i-1)(\beta+1)+\alpha_k+\gamma+1]!} (x-a)^{(2^i-1)(\beta+1)+\alpha_k+\gamma+1}. \tag{4}$$

Formula (3) is especially convenient if it is required to find several solutions of equation (1) with different initial conditions and right-hand sides f, in particular if it is required to construct a fundamental system of solutions of the homogeneous equation $L[y] = 0$.

For the effective use of the proposed approximate method it is necessary to choose the functions $W_0(x, s)$ and $z(x)$ rationally and to estimate $|K^{(k)}(x, s) - W_0^{(k)}(x, s)|$. From the theorem stated and estimate (4) it follows that W_0 and z should preferably be chosen so that $\alpha_k > 0,\ \beta > 0,\ \gamma > 0$. The latter conditions will be satisfied if $W_0^{(k)}(s, s) = K^{(k)}(s, s),\ z^{(k)}(a) = u^{(k)}(a)$ for $n \leq k \leq n + j$, where j is some positive number. Indeed, from Taylor' s formula it follows that in this case $\alpha_k = n - k + j + 1$ $(k = 0, \dots, n-1)$, $\beta = j + 1,\ \gamma = j + 1$. For example, one may put

$$W_0(x, s) = \frac{(x-s)^{n-1}}{(n-1)!} + \frac{(x-s)^n}{n!}\, g_{n-1}(s)$$

or

$$W_0(x, s) = \frac{(x-s)^{n-1}}{(n-1)!} + \frac{(x-s)^n}{n!}\, g_{n-1}(s) + \frac{(x-s)^{n+1}}{(n+1)!}\left[g_{n-1}^2(s) + \frac{d}{ds} g_{n-1}(s) + g_{n-2}(s)\right].$$

To obtain an estimate of the quantity $|K^{(k)}(x, s) - W_0^{(k)}(x, s)|$, one may use the following considerations. Let

$$L_0[y] \equiv y^{(n)} - \sum_{k=0}^{n-1} r_k(x) y^{(k)},$$

where $r_k \geq |g_k|$ for $k = 0, \dots, n-1,\ x \in [a, b]$, and $K_0(x, s)$ is the Cauchy function of the operator $L_0[y]$. Then in the triangle $a \leq s < x \leq b$ the inequality $K_0^{(k)}(x, s) \geq |K^{(k)}(x, s)|$ holds. Hence, and from (2), we have:

$$|K^{(k)}(x, s) - W_0^{(k)}(x, s)| \leq Q \int_s^x K_0^{(k)}(x, t)(t-s)^\beta \, dt.$$

It should be noted that the construction of $K_0(x, s)$ for $r_k = \text{const}$ $(k = 0, \dots, n-1)$ reduces to algebraic operations, since $K_0(x, s) = Y(x - s)$, where $Y(x)$ is the solution of the equation

$$L_0[y] = 0, \qquad y^{(k)}(a) = \delta_{k,n-1} \quad (k = 0, \dots, n-1).$$

Another estimate can be obtained as follows. Since

$$\left|K^{(k)}(x,s) - W_0^{(k)}(x,s)\right| \leq$$

$$\leq Q\int_s^x \left|K^{(k)}(x,t) - W_0^{(k)}(x,t)\right| (t-s)^{\beta}\, dt + Q\int_s^x \left|W_0^{(k)}(x,t)\right| (t-s)^{\beta}\, dt$$

$$(a \leq s \leq x < b, \quad k \leq n-1),$$

then, by the theorem on the integral inequality (1), we have:

$$\left|K^{(k)}(x,s) - W_0^{(k)}(x,s)\right| \leq (-1)^{\beta} Q \int_s^x |W_0(x,t)|\; v(s-t)\, dt,$$

where $v(x)$ is the solution of the equation

$$y^{(\beta+1)} = (-1)^{\beta+1} Q y, \qquad y^{(k)}(a) = \delta_{k,n-1} \quad (k = 0, \dots, n-1).$$

In particular, for $\beta = 1$,

$$\left|K^{(k)}(x,s) - W_0^{(k)}(x,s)\right| \leq \sqrt{Q} \int_s^x \operatorname{sh} \sqrt{Q}(t-s) \left|W_0^{(k)}(x,t)\right| dt.$$

In conclusion, let us note some properties of the sequence considered, which are convenient to use in estimating the limits of applicability of Chaplygin' s theorem on a differential inequality [2,4] and in studying questions of existence and uniqueness of the solution of certain boundary-value problems connected with these estimates [4,5].

Denote by $a \leq s \leq x < \Delta^k$ the largest of the triangles $a \leq s \leq x < \delta$ in which $K^{(k)}(x,s) \geq 0$ $(k = 0, \dots, n-1)$, and by $a \leq s \leq x < \Delta_i^k$ the largest of the triangles $a \leq s \leq x < \delta_i$ in which $W_i^{(k)}(x,s) \geq 0$. On the basis of the relations

$$\varphi_{i+1}(x,s) = -\int_s^x \varphi_i(x,t)\varphi_i(t,s)\, dt,$$

where $\varphi_i(x,s) = L[W_i(x,s)]$, and

$$W_i(x,s) = K(x,s) + \int_s^x K(x,t)\varphi_i(t,s)\, dt,$$

which are easily verified, one can prove the following assertion.

Theorem 2. Suppose that, for some m, in the triangle $a \leq s \leq x \leq b$ the residual $\varphi_m(x,s)$ does not change sign. Then, for $i > m$, if $\varphi_m \geq 0$, and for $i > m-1$, if $\varphi_m \leq 0$, the inequalities $\varphi_i \leq 0$, $a < \Delta_i^k \leq \Delta_{i+1}^k \leq \Delta^k$, $W_i^{(k)} \leq W_{i+1}^{(k)} \leq K^{(k)}$ $(a \leq s \leq x < \Delta_i^k,\ k = 0, \dots, n)$ hold.

From Theorem 2 there follows directly the following criterion for the positivity of the Cauchy function [3,5]:

In order that, in a given triangle $a \leq s < x < b$, the inequality $K^{(k)}(x,s) > 0$ $(k \leq n)$ hold, it is necessary and sufficient that in this

in the triangle there exists a function $W(x,s)$, n times continuously differentiable with respect to x, such that $W^{(j)}(s,s) = \delta_{j,n-1}$ $(j = 0, \dots, n-1)$; $W^{(k)}(x,s) > 0$, and $L[W(x,s)] \leq 0$ for $a \leq s < x < b$.

Theorem 2 and the extraordinary convergence of the sequence $\{W_i\}$ make it possible in practice to obtain a lower estimate for Δ^k. In this connection it is convenient to put $W_0(x,s) = K_1(x,s)$ or $W_0(x,s) = K_2(x,s)$, where $K_j(x,s)$

$$(j = 1, 2)$$

is the Cauchy function of the operation

$$L_j[y] \equiv y^{(n)} - \sum_{k=0}^{n-1} p_{kj}(x) y^{(k)}.$$

$$(p_{k1} \leq g_k \leq p_{k2}, \quad k = 0, \dots, n-1;\ x \in [a,b]).$$

A sequence analogous to the one considered can be constructed for systems of ordinary differential equations, for certain classes of partial differential equations, and, in general, for those differential equations in Banach spaces whose solution can be represented in the form

$$y(x) = K(x,a)y(a) + \int_a^x K(x,s) f(s)\, ds.$$

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Note: Figure translations are in progress. See original paper for figures.